Bost-Connes type dynamics and arithmetic of function fields

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Declaration

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Abstract

The work of this thesis falls within two main themes: the analogy between number fields and function fields of an algebraic curve over a finite field; and the recovery of arithmetic information from the Bost-Connes system associated to such global fields. We present a new construction for Bost-Connes systems for function fields based on the ring of S-integers and show that it provides a suitable framework to generalize the work of Takeishi to the function field case and thus recover class numbers. Further, we investigate an analogue of the Deligne-Ribet topological dynamical system that appears in Bost-Connes systems and show that isomorphisms that respect these dynamics are equivalent to arithmetic equivalences of the related global fields.

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17And I gave my heart to know wisdom, and to know
madness and folly: I perceived that this also is vexation
of spirit. 18For in much wisdom is much grief: and he
that increaseth knowledge increaseth sorrow.

(Ecclesiastes 1:17–18)

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Chapter 1

Introduction

It is well know that there is a close analogy between the theory of number fields and that of algebraic curves over finite fields; the analogy is so close that we often use the term *global field* to refer to both classes. This thesis studies this analogy in the specific example of Bost-Connes systems. A Bost-Connes system for $\mathbb{Q}$ was first described in [3] as a C^*-dynamical system $(\mathcal{A}_{\mathbb{Q}}, \sigma_{t,\mathbb{Q}})$ whose properties exhibited a variety of arithmetic data, such as the Riemann ζ-function or the maximal abelian extension of $\mathbb{Q}$. Since then, analogous systems have been described for imaginary quadratic number fields in [7], and then for a general number field in [19], [24], and [38]. With the results of the [38] in particular, the study of Bost-Connes systems associated to number fields has reach a level of maturity that, for varying levels of explicitness, one can construction a Bost-Connes system for any number field K such it exhibits the corresponding arithmetic data from the same properties as does the system for $\mathbb{Q}$ in [3]. There is less certainty about the appropriate C^*-dynamical system associated to a function field K over a finite field $\mathbb{F}_q$, however. In [9] the authors defined a dynamical system associated to a function field that starts with a groupoid built from an equivalence relation of commensurability on a space of lattices, similar to the number field construction in [6]. One key innovation is that the algebra resulting from their construction took values in a field of characteristic p rather than $\mathbb{C}$. Whereas the construction in [22] resulted in a C^*-algebra. It similarly starts with an quotient space, but the underlying space is characters on the set of torsion points of a Drinfeld module, which are known to recover an explicit set of generators for abelian extensions of a function field. Further, the author of [30] defined two dynamical systems, one a C^*-dynamical system with values in $\mathbb{C}$ and the other with values in a field of characteristic p. Both systems begin with an action of the fractional ideals J on a topological monoid ${}^{\mathbb{A} \times G}/_{\hat{\mathcal{O}}^*}$ composed of the adele ring $\mathbb{A}$, an abelian Galois group G and a group of units lying in the ideles $\hat{\mathcal{O}}$. This is similar to the "modern" number theory approach taken in, exempli gratia, [38].

Once one has an acceptable definition for a Bost-Connes system, it is natural to ask:

Question. *What properties do global fields K and L share given an equivalence (exempli gratia, $\mathbb{R}$-equivariant Morita equivalence, isomorphism, et cetera) of the C^*-dynamical systems $(\mathcal{A}_K, \sigma_{t,K})$ and $(\mathcal{A}_L, \sigma_{t,L})$?*

One attempt to answer this question in the number field case is in [12], wherein the authors showed that one has an isomorphism of number fields if one has an $\mathbb{R}$-equivariant isomorphism of dynamical systems that also preserves a specific sub-algebra. It has been shown in [32] and [33] that other number theoretic information, such as narrow class number or the Dedekind ζ-function, can be recovered from the C^*-algebra without the dynamics. Another approach to this question is to ask not about equivalences of the C^*-dynamical system but instead to ask about the dynamics of the fractional ideals J on the topological monoid $\mathbb{A} \times G / \hat{\mathcal{O}}^*$. This approach was taken for the function field case in [10] and for both classes of global fields in [13] and [11]. In these cases it was shown that equivalences of the topological dynamical system led to isomorphisms of the global fields, though the proof for an isomorphism of a function field is different to that of a number field, showing where the analogy between the two breaks down.

It is important to note that the study of Bost-Connes systems and their related objects has yielded new information about number theory. In particular the results of [12] and [13] allowed one to state a purely number-theoretic condition (an isomorphism of the group of Dirichlet characters that preserves L-functions) for when two global fields are isomorphic.

In this thesis we will study a special case of the the C^*-dynamical system in [30] for algebraic curves over a finite field. This particular case has not been studied before and we show that we can recover arithmetic information from it. At the start of our study we began by attempting to find an analogous result to [32], wherein the author recovered the narrow class number of a number field from the C^*-algebra of its Bost-Connes system. Since the narrow class number results from excluding the real Archimedean primes, we posited that we must exclude some of the primes of our function field. This means that our constructions are associated to only a function field K and a non-empty, finite set of primes S of K. The primes of S then "chose" for us a model of a maximal abelian extension $K^{\mathrm{ab}}(S)$. While $K^{\mathrm{ab}}(S)$ is known to be smaller than the maximal abelian extension, it turns out to be appropriate for recovering the S-class number from the class of C^*-algebra, as stated in Theorem 4.1. If we include the time-evolution in our study, we are also able to recover the group of Principal S-divisors and the S-divisor norm in Theorem 4.2. Though we had constructed $K^{\mathrm{ab}}(S)$ so that we could prove Theorem 4.1, it has proven to be a good model: it allows us to prove Lemma 3.1.5, which means that characters of the units of a local field $K_{\mathfrak{p}}$ can be used to construct a compatible character of the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$. We exploited this property when we turned to study the dynamics of the underlying topological monoid without the C^*-algebra structure. In doing so we proved Theorem 5.1: that equivalences of the dynamics of the underlying monoid are themselves equivalent to analogues of the arithmetic equivalences in [13] and [11]. While we are unable to recover an isomorphism of function fields in this thesis, the "failure" to show an isomorphism echos the failure of the function-field methods in [13] to prove the number field isomorphism, which we describe in more detail in Chapter 6.

Despite the strong analogy between global function fields and number fields, a number of stark differences remain, which required the development of new techniques used to prove our results. The bulk of these differences come from Class Field Theory. Let K be a global field and $\vartheta_K : \mathbb{A}_K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the Artin Reciprocity map. When K is a number field it is known that $\ker \vartheta_K = K^* \cdot \overline{\mathcal{O}_+^*(K)}$, where $\mathcal{O}_+^*(K)$ is the group of totally positive units of K embedded in the ideles. But when K is a global function field we have that $\ker \vartheta_K = K^*$. This required us to construct a specific quotient of the maximal abelain Galois group in order to prove Proposition 4.1.1 and thus Theorem 4.1. Moreover, for a number field ϑ_K is surjective, while for a function field we have that $\vartheta_K(\mathbb{A}_K^*)$ is dense in the Galois group. This introduced a number of technical difficulties that are resolved in Section 2.2 and Chapter 3. In particular we had to show that our quotient resulted in an infinite extension by studying the closed, but not open, subgroups of the dense image of ϑ_K (Corollary 2.2.14). Further, we also needed to prove Lemma 3.1.5, which generalizes the Grünwald-Wang Theorem (Theorem 5 of Chapter X of [1]), to show that we can construct characters of quotient of the maximal abelian Galois group from characters of the multiplicative group of a local field. This was instrumental in proving Theorem 5.1, especially Lemma 5.3.1 and Proposition 5.3.9. One should also note that the norm of a divisor of a function field behaves differently than that the analogous norm of an ideal of a number field. For a function field we have that $N(D) = q^{\deg D}$, which immediately tells us that the image of the norm has only a single generator. But since the degree of principal divisors is always 0, this also means that the image of the norm of principal divisors is trivial, unlike the number field case. This greatly affects Theorem 4.2 which relies on a non-trivial image. Hence the use of the principal S-divisors and the S-norm. It is noteworthy that by excluded the primes in S we are able to proof the result and recover more arithmetic information than we would be able to otherwise.

The thesis will proceed as follows: In Chapter 2 we will introduce a plethora of preliminary results about function fields, their class field theory, and C^*-algebras. We introduce our construction of a Bost-Connes system associated to a function field K with finite, non-empty set of primes S in Chapter 3. In Chapter 4 we show that our constructions have anagalous results to [32] for the S-class number and the dual of the group of principal S-divisors. In Chapter 5 we study the dynamics on the topological monoid $\mathbb{A} \times G /_{\hat{\mathcal{O}}^*}$ from our construction and show that it can recover the same arithmetical properties as in [13] and [11].

Chapter 2

Preliminaries

The goal of this chapter is to provide basic facts *and notation* for working with global function fields and C^*-dynamical systems. In particular, we will define and prove properties about the ring of S-integers, the S-class group, et cetera, for a global function field in Section 2.1. This section presents a summarized version of the necessary material from Chapters 5, 9, and 14 of [29]. In Section 2.2 we will state basic results from Class Field Theory and prove some results particular to function fields. This material comes from [27], [5] and [1], along with a discussion of the Weil group inspired by [8]. We will close in Section 2.3 by describing background topics in C^*-algebras, in particular crossed products, primitive ideals, induced representations, and Morita equivalences. This last section on C^*-algebras contains material found in [15], [36] and [28].

2.1 Function Field Arithmetic

$\mathbb{F}_q$ denotes a finite field with $q = p^n$ elements, where p is prime. We will use K or L to denote a global function field, by which we mean a finite extension of $\mathbb{F}_q(t)$. One of the key differences between a global function field and a number field is that the former lacks a ring of integers. Even in the rational function field $\mathbb{F}_q(t)$, a choice of different indeterminate, say $1/t$, results in different rings $\mathbb{F}_q[t]$ and $\mathbb{F}_q[\frac{1}{t}]$. Consequently, we do not have a model of the ring of integers that can describe the whole set of integral prime ideals. Thus with function fields we instead define a *prime* $\mathfrak{p}$ to be a equivalence class of topologically equivalent valuations. We will use $\mathcal{S}_K$ to denote the set of all such primes. Given a specific prime $\mathfrak{p}$ we will use $|\cdot|_{\mathfrak{p}}$ to denote the associated absolute value, $K_{\mathfrak{p}}$ for the completion of K under this absolute value, and $v_{\mathfrak{p}}$ to denote a normalized *exponential* valuation. Additionally, we will use $\mathcal{O}_{\mathfrak{p}}(K)$ $= \{f \in K : v_{\mathfrak{p}}(f) \geq 0\}$ for the valuation ring at $\mathfrak{p}$ inside K, and $\mathcal{O}_{\mathfrak{p}}$ for the valuation ring sitting inside the completion $K_{\mathfrak{p}}$. The nonzero ideals of $\mathcal{O}_{\mathfrak{p}}(K)$ can be written as $\pi_{\mathfrak{p}}^n \mathcal{O}_{\mathfrak{p}}(K)$, where $\pi_{\mathfrak{p}}$ is some element of $\mathcal{O}_{\mathfrak{p}}(K)$ with $v_{\mathfrak{p}}(\pi_{\mathfrak{p}}) = 1$ and $n \in \mathbb{N}$.

Let $\mathcal{D}(K)$= $\bigoplus_{\mathfrak{p}} \mathbb{Z} = \{D = \sum_{\mathfrak{p}} D_{\mathfrak{p}}\mathfrak{p} : D_{\mathfrak{p}} \in \mathbb{Z}\}$ be the Divisor group of K, that is, the free abelian group generated by the set of primes. For each $f \in K^*$, we define div(f)= $\sum_{\mathfrak{p}} v_p(f)\mathfrak{p}$ to be the principal divisor of f. We use $\mathcal{P}(K)$ to denote the image of div and $\mathcal{C}l(K)$= $\mathcal{D}(K)/\mathcal{P}(K)$ the divisor class group.

We define the degree of a prime deg$(\mathfrak{p})$ to be the degree of the extension of the residue class field of the valuation ring $\mathcal{O}_{\mathfrak{p}}$ of $\mathfrak{p}$ over $\mathbb{F}_q$. We may extend this to a divisor D by deg$(D) := \sum_{\mathfrak{p}} D_{\mathfrak{p}}deg(\mathfrak{p})$. We denote the set of degree 0 divisors by $\mathcal{D}^0(K)$. In fact, for any subgroup $\mathcal{G}$ of $\mathcal{D}(K)$ (or, indeed, any quotient $\mathcal{G} = \mathcal{D}(K)/\mathcal{H}$ where all divisors in $\mathcal{H}$ have degree 0), we will denote by $\mathcal{G}^0$ the subgroup of degree 0 divisors within $\mathcal{G}$. Since the principal divisors have degree 0, this includes $\mathcal{C}l^0(K)$.

We should note that these objects are not analogues of the group of fractional ideals, principal ideals, et cetera, in a number field; the group of fractional ideals excludes the "primes at infinity". But in a global function field all primes are non-Archimedean and we therefore include all primes. The analogous objects in number theory are the Arakelov (or *replete*) divisors. We note that $\mathcal{C}l(K)$ is not finite in general (though $\mathcal{C}l^0(K)$ is). We will simulate this exclusion of primes at infinity by arbitrarily nominating a non-empty finite set of primes S to exclude themselves from our definitions. We then define the group of S-divisors $\mathcal{D}_S(K)$ to be

$$\mathcal{D}_S(K) := \{D = \sum_{\mathfrak{p} \notin S} D_\mathfrak{p}\mathfrak{p} : D_\mathfrak{p} \in \mathbb{Z}\}$$

Similarly we also define the set of principal S-divisors $\mathcal{P}_S(K)$ to be the image of the map $\mathrm{div}_S(f) = \sum_{\mathfrak{p} \notin S} v_\mathfrak{p}(f)\mathfrak{p}$. We then can define the S-class group:

$$\mathcal{C}l_S(K) := \mathcal{D}_S(K)/\mathcal{P}_S(K)$$

Last but not least we have the ring of S-integers $\mathcal{O}_S(K)$

$$\mathcal{O}_S(K) := \{a \in K : v_\mathfrak{p}(a) \geq 0 \; \forall \mathfrak{p} \notin S\}$$

We will show that $\mathcal{O}_S(K)$ and company are the appropriate analogues of the ring of integers, class group, et cetera. In particular, we will show that the units of $\mathcal{O}_S(K)$ are given by elements with zero valuation on primes outside of

S, that it is a Dedekind domain and its primes are in bijective correspondence with $\mathcal{S}_K \setminus S$, and that $\mathcal{C}l_S(K)$ is finite. These results are well-known and can be found in [27] or [29].

Proposition 2.1.1. *The unit group $\mathcal{O}_S^*(K)$ of $\mathcal{O}_S(K)$ is given by*

$$E(S) := \{a \in K^* : v_{\mathfrak{p}}(a) = 0 \ \forall \mathfrak{p} \notin S\}$$

Moreover, $\mathcal{O}_S^*(K)/\mathbb{F}_q^*$ is a finitely generated free group of rank at most $\#S - 1$.

Theorem 2.1.2. *$\mathcal{C}l_S(K)$ is group isomorphic to $\mathcal{C}l(\mathcal{O}_S(K))$, the ideal class group of $\mathcal{O}_S(K)$, and is finite. We denote the number of elements of this group by h_S.*

The proof of the above requires some additional results and notation, so first let us continue collecting some facts about this ring.

Now, let $\mathcal{D}^S(K)$ denote the subgroup of divisors generated only by primes in S, and $\mathcal{P}^S := \mathcal{P}(K) \cap \mathcal{D}^S(K)$. The result we need is

Proposition 2.1.3. *The following sequences are exact:*

$$0 \to \mathbb{F}_q^* \hookrightarrow E(S) \to \mathcal{P}^S \to 0$$

$$0 \to \mathcal{D}^{S^0}(K) \Big/ \mathcal{P}^S(K) \hookrightarrow \mathcal{C}l^0(K) \to \mathcal{C}l_S(K) \to C_{\frac{d}{i}} \to 0$$

where d is the greatest common factor of the elements in the set $\{\deg \mathfrak{p} : \mathfrak{p} \in S\}$; i is the greatest common factor of the elements in the set $\{\deg \mathfrak{p} : \mathfrak{p} \in \mathcal{S}_K\}$; and $C_{\frac{d}{i}}$ is the cyclic group of order d/i.

Proof. For the first sequence, the map $E(S) \to \mathcal{P}^S(K)$ (which takes an alleged

S-unit to its principal divisor) is surjective. Moreover, if an element u is in its kernel then it must have $v_{\mathfrak{p}}(a) = 0\ \forall \mathfrak{p} \in \mathcal{S}_K$, id est, $a \in \mathbb{F}_q^*$.

For the second sequence, first note that if $D + \mathcal{P}^S(K)$ is a non-trivial class in $D^{S^0}(K) \Big/ \mathcal{P}^S(K)$, then $D \notin \mathcal{P}(K)$, and so

$$\ker\left(\mathcal{D}^{S^0}(K) \Big/ \mathcal{P}^S(K) \hookrightarrow \mathcal{C}l^0(K)\right) = 0.$$

Now let $\tau : \mathcal{D}(K) \to \mathcal{D}_S(K)$ be given by

$$\tau(D) = \sum_{\mathfrak{p} \notin S} v_{\mathfrak{p}}(D)\mathfrak{p}.$$

Since $\tau(\mathcal{P}(K)) = \mathcal{P}_S(K)$ it induces a map $\tilde{\tau} : \mathcal{C}l(K) \to \mathcal{C}l_S(K)$. Clearly $\ker\tau = \mathcal{D}^S(K)$, which implies that

$$\ker\tilde{\tau} = \left(\mathcal{D}^S(K) + \mathcal{P}(K)\right) / \mathcal{P} \cong \mathcal{D}^S(K) \Big/ \mathcal{P}^S(K).$$

Restricting to divisors of degree 0 this gives that

$$\mathrm{Im}\left(\mathcal{D}^{S^0}(K) \Big/ \mathcal{P}^S(K) \hookrightarrow \mathcal{C}l^0(K)\right) = \ker(\tilde{\tau} : \mathcal{C}l^0(K) \to \mathcal{C}l_S(K)).$$

The same map also gives us an isomorphism between $\mathcal{C}l_S(K)$ and $\mathcal{D}(K) \Big/ (\mathcal{P}(K) + \mathcal{D}^S(K))$. Since principal divisors all have degree 0, we also know that the following composition (not exact sequence!) is surjective:

$$\upsilon : \mathcal{C}l_S(K) \to \mathcal{D}(K) \Big/ (\mathcal{P}(K) + \mathcal{D}^S(K)) \to \mathcal{D}(K) \Big/ (\mathcal{D}^0(K) + \mathcal{D}^S(K)) \to C_{\frac{d}{i}}$$

The last map is given by the degree map: by definition the image of the degree map in $\mathbb{Z}$ is the ideal $i\mathbb{Z}$, whereas $\deg(\mathcal{D}^0(K)+\mathcal{D}^S(K)) = d\mathbb{Z}$. Hence the image of the penultimate group under the degree map is $i\mathbb{Z}/d\mathbb{Z} \cong C_{\frac{d}{i}}$. The kernel of this map is the classes of divisors with degree 0, hence $\mathrm{Im}(\tilde{\tau} : \mathcal{C}l^0(K) \to \mathcal{C}l_S(K)) = \ker(\upsilon : \mathcal{C}l_S(K) \to C_{\frac{d}{i}})$. □

Theorem 2.1.4. *$\mathcal{O}_S(K)$ is a Dedekind domain and its non-zero prime ideals are in bijective correspondence with the primes of K that are not in S.*

Proof. Assume that we can construct an element $x \in K^*$ such that $v_{\mathfrak{p}}(x) < 0$ when $\mathfrak{p} \in S$. (We will show that this assumption is true later.) Let R be the integral closure of $\mathbb{F}_q[x]$ in K. It is known (in say, [39] Chapter V, Theorem 19) that R is a Dedekind domain.

Now let $\mathfrak{p} \notin S$. Then $x \in \mathcal{O}_{\mathfrak{p}}(K)$, and so $R \subseteq \mathcal{O}_{\mathfrak{p}}(K)$. Consider the ideal $\mathfrak{q} := (\pi_{\mathfrak{p}}) \cap R$. This ideal must be nonzero, as otherwise then K (which is the quotient field of R) would map injectively into the finite residue field $\mathcal{O}_{\mathfrak{p}}(K)/(\pi_{\mathfrak{p}})$. So then $\mathfrak{q}$ must be a maximal ideal of R. Naively, its localization $R_{\mathfrak{q}} \subset K$. But every element of $R_{\mathfrak{q}}$ is of the form $\frac{r}{s}$ where $r \in R$ and $s \in R\backslash\mathfrak{p}$. So we have that $v_{\mathfrak{p}}(r) \geq 0$ and $v_{\mathfrak{p}}(s) \leq 0$, id est $v_{\mathfrak{p}}(\frac{r}{s}) \geq 0$ and $R_{\mathfrak{q}} \subset \mathcal{O}_{\mathfrak{p}}(K)$. But since discrete valuation rings are maximal subrings, we have that $R_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{p}}(K)$.

Going the other way, if $\mathfrak{q}$ is a maximal ideal of R, then $R_{\mathfrak{q}}$ is a discrete valuation ring and $(\mathfrak{q}, R_{\mathfrak{q}})$ is a discrete valuation ring of K containing x. Thus the mapping

$$\mathfrak{q} \mapsto (\mathfrak{q}, R_{\mathfrak{q}})$$

is a bijection between the primes of K not contained in S and the prime ideals of R. To finish, using the fact that R is a Dedekind domain along with Section

10.4 of [23] we have that

$$R = \bigcap_{\mathfrak{q} \subset R} R_{\mathfrak{q}} = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}}(K) = \mathcal{O}_S(K)$$

To construct such an x, label the primes of S as $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and let M be an positive number. Consider the vector spaces over $\mathbb{F}_q$:

$$L(M\mathfrak{p}_i) = \{x \in K^* : v_{\mathfrak{p}}(\operatorname{div}(x) + M\mathfrak{p}_i) \geq 0 \; \forall \; \mathfrak{p}\} \cup \{0\}$$

Denote their dimensions by $l(M\mathfrak{p}_i)$. By the Riemann-Roch Theorem, we know that we can choose an $\tilde{M}$ large enough so that $l(\tilde{M}\mathfrak{p}_i) = \tilde{M} \deg \mathfrak{p}_i - g + 1$, where g is genus of the field K. Then we have that $L(\tilde{M}\mathfrak{p}_i) \subset L((\tilde{M}+1)\mathfrak{p}_i)$. Choose $x_i \in L((\tilde{M}+1)\mathfrak{p}_i) \setminus L(\tilde{M}\mathfrak{p}_i)$ for each $\mathfrak{p}_i \in S$. Each x_i has $v_{\mathfrak{p}_i}(x_i) = -\tilde{M} - 1$ and $v_{\mathfrak{p}}(x_i) \geq 0$ for a $\mathfrak{p} \neq \mathfrak{p}_i$. Let $x = x_1 x_2 \ldots x_s$. $\square$

The previous results about global function fields now follow easily:

Proof. **Of Proposition 2.1.1** By Proposition 2.1.3 we have that $E(S)/\mathbb{F}_q^* \cong \mathcal{P}^S(K)$, which is a subgroup of the free group $\mathcal{D}_S^0(K)$ on $\#S - 1$ generators. By the proof of Proposition 2.1.4 we also have

$$\mathcal{O}_S^*(K) = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}}^*(K)$$

Which gives us $\mathcal{O}_S(K) = E(S)$. $\square$

Proof. **Of Theorem 2.1.2**

The previous proposition gives us a bijection between the generators of the the S-class group of K and the ideal class group of $\mathcal{O}_S(K)$, and so a group isomorphism directly follows. We show that they are also finite:

From Proposition 2.1.3 we see that $\mathcal{C}l_S(K)$ is finite if $\mathcal{C}l^0(K)$ is finite. Use the notation for L and l as in the proof of Proposition 2.1.4 and let g be the genus of K. Let D be a divisor of degree 1 and A a divisor of degree 0. Then $\deg(gD + A) = g$ so by the Riemann-Roch theorem $l(gD + A) \geq 1$. If $f \in L(gD + A)$ then their exists an effective divisor $B = \operatorname{div}(f) + gD + A$ of degree g and such that $A \sim B - gD$. Thus the number of divisor classes of degree zero is bounded above by the number of effective divisors of degree g.

Any global function field K is an extension of a rational function field $\mathbb{F}_q(x)$. It is well known that there are only finitely many primes in an extension $K/\mathbb{F}_q(x)$ lying above a specific prime of $\mathbb{F}_q(x)$. Therefore if $\sum_{\mathfrak{p}} a_{\mathfrak{p}}\mathfrak{p}$ is an effective divisor of degree g, each prime $\mathfrak{p}$ can have at most degree g, and thus there are only finitely many divisors of degree g. $\square$

Since Theorem 2.1.4 tells us that the primes of $\mathcal{O}_S(K)$ are in bijective correspondence with $\mathcal{S}_K \setminus S$ we can define an S-Norm N_S on S-divisors:

$$N_S(D) := \#\left(\mathcal{O}_S(K) \Big/ \prod_{\mathfrak{p}\in\operatorname{supp}(D)} \pi_{\mathfrak{p}}^{D_{\mathfrak{p}}}\right)$$

Clearly we still have $N_S(D_1+D_2) = N_S(D_1)N_S(D_2)$. This S-norm agrees with the divisor norm

$$N(D) = q^{\deg(D)}$$

when it is restricted to S-divisors.

2.2 Class Field Theory

Following [27], we will use $\mathbb{A}(K)$ to denote the topological ring of adeles

$$\mathbb{A}(K) := \prod_{\mathfrak{p}\in\mathcal{S}_K}{}' K_{\mathfrak{p}} = \{(a_{\mathfrak{p}}) : a_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p}\}.$$

We will use the term **support of** $(a_{\mathfrak{p}})$ to denote the set of primes $\{\mathfrak{p} : a_{\mathfrak{p}} \neq 0\}$. The topology is the *restricted product topology* which is generated by sets of the form

$$U_R = \prod_{\mathfrak{p}\in R} U_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in\mathcal{S}_K\setminus R} \mathcal{O}_{\mathfrak{p}}$$

for finite subsets $R \subset \mathcal{S}_K$. It is easy to verify that point-wise addition and multiplication are continuous in this topology. We denote two important subspaces of the adeles: the first is $\hat{\mathcal{O}}(K)$, its maximal compact sub-ring:

$$\hat{\mathcal{O}}(K) := \prod_{\mathfrak{p}\in\mathcal{S}_K} \mathcal{O}_{\mathfrak{p}}.$$

The second is the diagonal embedding $K \hookrightarrow \mathbb{A}(K)$ by $f \mapsto (a_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{S}_K}$ where $a_{\mathfrak{p}} = f$ for all $\mathfrak{p}$. We will typically treat any $f \in K$ as implicitly being in $\mathbb{A}(K)$ as well. The image of this embedding is the set of *principal adeles.*

Similarly we also use $\mathbb{A}_S(K)$ to denote the topological ring of S-adeles of K

$$\mathbb{A}_S(K) = \prod_{\mathfrak{p}\in\mathcal{S}_K\setminus S}{}' K_{\mathfrak{p}} = \{(a_{\mathfrak{p}})_{p\in\mathcal{S}_K - S} : a_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p}\}$$

Its topology is defined in the same manner as for $\mathbb{A}(K)$. It also has $\hat{\mathcal{O}}_S$ as its maximal subring

$$\hat{\mathcal{O}}_S := \prod_{\mathfrak{p}\in\mathcal{S}_K\setminus S} \mathcal{O}_{\mathfrak{p}}$$

and a diagonal embedding $K \hookrightarrow \mathbb{A}_S(K)$, we denote these principal S-adeles by K_S too differentiate them from the principal adeles. The following theorem is a standard, but important, result for understanding the topology of the principal adeles:

Theorem 2.2.1. *(Strong Approximation Theorem [5], pg 67.) Let $\mathfrak{p} \in \mathcal{S}_K$ and let $S = \{\mathfrak{p}\}$. Then K_S is dense in $A_S(K)$.*

We will use $\mathbb{A}^*(K)$ (the idele group), $\mathbb{A}_S^*(K)$ (the S-idele group), et cetera, to denote the unit group of $\mathbb{A}(K)$, $\mathbb{A}_S(K)$, et cetera, respectively. The topology on the idele group is not the same as the subspace topology from $\mathbb{A}(K)$, rather it is the restrict product topology whose basis sets are of the form

$$U_R = \prod_{\mathfrak{p} \in R} U_{\mathfrak{p}} \times \prod_{\mathfrak{p} \in \mathcal{S}_K \setminus R} \mathcal{O}_{\mathfrak{p}}^*$$

where $U_{\mathfrak{p}}$ is an open subgroup of $K_{\mathfrak{p}}^*$.

Let *ad*: $\mathbb{A}_S^*(K) \to \mathcal{D}_S(K)$ denote the map

$$(a_{\mathfrak{p}}) \mapsto \sum_{p \notin S} v_{\mathfrak{p}}(a_{\mathfrak{p}})\mathfrak{p}$$

Clearly this map is surjective. Its kernel is exactly described by vectors a such that $v_{\mathfrak{p}}(a) = 0 \ \forall \mathfrak{p} \notin S$, so that we have:

Proposition 2.2.2. $\mathcal{D}_S(K) \cong \mathbb{A}_S^*(K) \Big/ \hat{\mathcal{O}}_S^*(K)$.

Definition 2.2.3. *A* ***standard split*** *for the map ad is a partial inverse $s : \mathcal{D}_S(K) \to \mathbb{A}_S^*(K)$ such that $s(\mathfrak{p}) = (\dots, 1, \pi_{\mathfrak{p}}, 1, \dots)$, where $\pi_{\mathfrak{p}}$ is a uniformizer for $\mathcal{O}_{\mathfrak{p}}(K)$.*

For any set R we will, by a slight abuse of notation, use $i : \mathbb{A}_S^* \to \mathbb{A}^*$ to denote the map

$$(a_\mathfrak{p})_{\mathfrak{p}\in\mathcal{S}_K - R} \mapsto (b_\mathfrak{p})_{\mathfrak{p}\in\mathcal{S}_K} \quad b_\mathfrak{p} = \begin{cases} a_\mathfrak{p} & \text{if } \mathfrak{p} \notin R \\ 1 & \text{otherwise} \end{cases}$$

2.2.1 The Artin Map

Let L be a finite abelian Galois extension of K and set $G := \mathrm{Gal}(L/K)$. Also let $\mathfrak{p}$ be a prime of K unramified in L and let $\mathfrak{q}$ be a prime of L over $\mathfrak{p}$. Recall that G acts on primes of L, for if $a \in L$ and $\sigma \in G$ then $|a|_{\sigma(\mathfrak{q})} := |\sigma^{-1}(a)|_\mathfrak{q}$ defines a new absolute value and thus a different prime also over $\mathfrak{p}$. The action of σ preserves Cauchy sequences so that $\sigma(\mathfrak{q}) : L_\mathfrak{q} \to L_{\sigma(\mathfrak{q})}$ is a $K_\mathfrak{p}$-isomorphism of the local fields.

The decomposition group of the prime $\mathfrak{q}$ is the subgroup of G

$$G_\mathfrak{q} := \{\sigma \in G : \sigma(\mathfrak{q}) = \mathfrak{q}\}$$

It is known (in say [5]) that for *any* prime $\mathfrak{p}$ of K, not just those unramified in L, we have

Proposition 2.2.4. *(Chapter VII, Prop 1.2 of [5]) There is an isomorphisms of finite abelian groups*

$$G_\mathfrak{q} \cong Gal(L_\mathfrak{q}/K_\mathfrak{p}) \cong Gal(L(\mathfrak{q})/K(\mathfrak{p}))$$

where $L(\mathfrak{q})$ and $K(\mathfrak{p})$ denote the residue class field of L (respectively, K) with respect to $\mathfrak{q}$ (respectively, $\mathfrak{p}$).

Since the Galois groups of finite field extensions are cyclic, the group $\mathrm{Gal}(L(\mathfrak{q})/K(\mathfrak{p}))$ has a canonical generator: the Frobenius automorphim $x \mapsto x^{\#K(\mathfrak{p})}$. The corresponding element of the group $G_{\mathfrak{q}}$ is the unique element $\mathrm{Frob}_{L/K}(\mathfrak{p}) \in \mathrm{Gal}(L/K)$ characterized by the property

$$\mathrm{Frob}_{L/K}(\mathfrak{p})(x) \equiv x^{\#K(\mathfrak{p})} \mod (\pi_{\mathfrak{q}}) \text{ for all } x \in \mathcal{O}_{\mathfrak{q}}.$$

Let us justify the notation $\mathrm{Frob}_{L/K}(\mathfrak{p})$: Note that if $\tau \in G$ we have that $G_{\tau(\mathfrak{q})} = \tau G_{\mathfrak{q}} \tau^{-1}$. Since the primes lying about $\mathfrak{p}$ in L are conjugate we know that $\tau(q)$ also lies above $\mathfrak{p}$, so that $\mathfrak{p}$ determines a conjugacy class $\mathrm{Frob}_{L/K}(\mathfrak{p})$ in G. But we also assume that L is an abelian extension, so that conjugacy class is a single element. Altogether we have a map:

$$\mathrm{Frob}_{L/K} : \{\mathfrak{p} \in \mathcal{S}_K : \mathfrak{p} \text{ unramified in } L\} \to \mathrm{Gal}(L/K).$$

If S_L denotes the finite set of primes $\mathfrak{p} \in \mathcal{S}_K$ that are ramified in L then it is easy to see that we can extend this map to a group homomorphism

$$\mathrm{Frob}_{L/K} : \mathcal{D}_{S_L}(K) \to \mathrm{Gal}(L/K).$$

So far the theory for global function fields and for number fields is the same, except that in the number field case we require that S_L also contains the Archimedean primes. The following theorem is the main standard result from class field theory for both forms of global fields.

Theorem 2.2.5. *(Main statements of class field theory, [5], pg 172.)*

A. *There exists a continuous group homomorphism $\vartheta_{L/K} : \mathbb{A}^*(K) \to Gal(L/K)$*

(called the Artin map for L/K) that satisfies

(i) $\vartheta_{L/K}$ *is continuous.*

(ii) $K^* \in \ker \vartheta_{L/K}$

(iii) $\vartheta_{L/K} \circ i(a_{\mathfrak{p}}) = \mathrm{Frob}_{L/K} \circ ad(a_{\mathfrak{p}})$ *for any* $(a_{\mathfrak{p}}) \in \mathbb{A}^*_{S_L}$.

B. *The Artin map* $\vartheta_{L/K}$ *is surjective and* $\ker \vartheta_{L/K} = K^* \cdot N_{L/K}(\mathbb{A}^*(L))$ *where* $N_{L/K} : \mathbb{A}_L \to A_K$ *is the norm of the field extension.*

C. *If* $K \subset L \subset M$ *is a tower of abelian extensions then we have a commutative diagram:*

$$\begin{array}{ccc} \mathbb{A}^*(K) \Big/ K^* \cdot N_{M/K}(\mathbb{A}^*(M)) & \xrightarrow{\vartheta_{M/K}} & Gal(M/K) \\ \downarrow & & \downarrow \\ \mathbb{A}^*(K) \Big/ K^* \cdot N_{L/K}(\mathbb{A}^*(L)) & \xrightarrow{\vartheta_{L/K}} & Gal(L/K) \end{array}$$

D. *For every open subgroup* N *of finite index in* $\mathbb{A}^*(K)/K^*$ *there exists a unique abelian extension* L/K *such that* $K^* \cdot N_{L/K}(\mathbb{A}^*(L)) = N$.

The commutative diagram in C. allows us to define a continuous group homomorphism ϑ_K with the maximal abelian extension of K:

$$\vartheta_K : \mathbb{A}^*(K) \to \varprojlim_{L} \mathrm{Gal}(L/K) \cong \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

where L runs through all finite abelian extensions of K.

Remark 2.2.6. *Note that there is also a local Artin map*

$$\vartheta_{K_{\mathfrak{p}}} : K^*_{\mathfrak{p}} \to Gal(K^{ab}_{\mathfrak{p}}/K_{\mathfrak{p}}).$$

Let L be a finite abelian extension of K and let $\mathfrak{q}$ be a prime of L lying above a prime $\mathfrak{p}$ in K. We already know from Prop. 2.2.4 that $Gal(L_\mathfrak{q}/K_\mathfrak{p})$ is isomorphic to a subgroup of $Gal(L/K)$. Since K^{ab} is the union of all finite abelian extensions L, we have an injective map $Gal(K_\mathfrak{p}^{ab}/K_\mathfrak{p}) \to Gal(K^{ab}/K)$. This results in the following commutative square:

$$\begin{array}{ccc} K_\mathfrak{p}^* & \xrightarrow{\vartheta_{K_\mathfrak{p}}} & Gal(K_\mathfrak{p}^{ab}/K_\mathfrak{p}) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \mathbb{A}^*(K) & \xrightarrow{\vartheta_K} & Gal(K^{ab}/K) \end{array}$$

We will now study the kernel and the image of the Artin map ϑ_K. Here the number field case and the global function field case diverge. Even so, it is well known that

Proposition 2.2.7. $\ker \vartheta_K = K^*$.

Proof. See Remark 5.5 of Chapter VII in [5], or alternatively [1], page 76. □

The image of ϑ_K is more complicated; this map is not surjective. To describe it will be helpful to define another arithmetic object: the Weil group. Note that the constant field of K^{ab} is $\overline{\mathbb{F}_q}$, and we know that $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$. This gives us a continuous surjective map $\pi : \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \hat{\mathbb{Z}}$ by restriction of the action of an automorphism to the constant field.

Definition 2.2.8. *The **Weil Group** is the topological group whose elements are $W := \pi^{-1}(\mathbb{Z}) \subset Gal(K^{ab}/K)$.*

This means that the Weil Group W consists of Galois automorphisms that, when restricted to $\overline{\mathbb{F}_q}$, act by integral powers of the Frobenius automorphism.

Our discussion of the Weil group is brief and is inspired by [8]. More details can also be found in [1] and some information on profinite groups can be found in [37].

Note. *It is more common to present a non-abelian version of Weil group, but it is not needed here.*

We consider $\ker \pi$ to have the subspace topology from $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and note that $\ker \pi$ is an open subgroup in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, but it is not open in W with respect to the subspace topology on W from $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. We will consider W with the following topology instead:

Proposition 2.2.9. *There exists a topological group structure on W such that $\ker \pi$ is open in W.*

Proof. Let U and $U' \subset \ker \pi$ be open neighbourhoods of the identity and $w \in W$. Then wU and wU' are open neighbourhoods of w. Such sets form a basis for a topology. To see this note any such U (respectively U') equals $\ker \pi \cap T$ (respectively $\ker \pi \cap T'$) for an open set T (respectively T') $\subset \mathrm{Gal}(K^{\mathrm{ab}}/K)$. If $x \in wU \cap w'U'$ and $V = \ker \pi \cap x^{-1}wT \cap x^{-1}w'T'$ then xV is a basis set and is contained in $wU \cap w'U'$. $\ker \pi$ is open by construction. We show that the group operations on W inherited from the Galois group are continuous under this topology. It is sufficient to demonstrate them for a neighbourhood of the identity. In particular, $\ker \pi$ is such a neighbourhood and multiplication and inversion are clearly continuous here, as it inherits a topological group structure. $\square$

We have a split exact sequence

$$1 \to \ker \pi \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\pi} \hat{\mathbb{Z}} \to 1$$

where $\hat{\mathbb{Z}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \,/_{\ker \pi}$ inherits the quotient topology from $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\ker \pi$ inherits the subspace topology from $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. Since π can be thus seen to be a quotient map we know that it is an open map. We use this fact in proving:

Proposition 2.2.10. *The injection of $\iota : W \hookrightarrow Gal(K^{ab}/K)$ is continuous with respect to the above topology and has dense image.*

Proof. It suffices to show continuity around an open neighborhood of the identity in W. One such neighborhood is $\ker \pi$, so continuity is a consequence of the previous proposition. To show that W is dense we will show that γU intersects W for any $\gamma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and U an open neighborhood of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ around the identity. Note that $\pi(U)$ is open and $\mathbb{Z}$ is dense in its profinite completion, so there is some $m \in \mathbb{Z}$ such that $m \in \pi(\gamma)\pi(U)$, say $m = \pi(\gamma u)$ for $u \in U$. It is clear that $\gamma u \in W \cap \gamma U$. $\square$

Theorem 2.2.11. *There is an inclusion-preserving bijection between open subgroups of finite index of $Gal(K^{ab}/K)$ and open finite index subgroups of W given by*

$$H \mapsto H \cap W \textit{ for } H \subset Gal(K^{ab}/K)$$

$$J \mapsto \bar{J} \textit{ for } J \subset W$$

Proof. Let H be an open subgroup of finite index in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. Since $W \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is continuous we have that $H \cap W$ is open in W. The map of the quotient spaces $W/H\cap W \to \mathrm{Gal}(K^{\mathrm{ab}}/K)/H$ is injective, so $H\cap W$ is of finite index, too. To see that $\overline{H\cap W} = H$, recall that W is dense in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, and so we see that $H\cap W$ is dense in H. Since H is open, it is also closed, and $\overline{H\cap W} = H$ follows.

Now let J be open and of finite index in W. We need to show that $\overline{J}\cap W = J$ and that $\overline{J}$ is of finite index (since $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is profinite we know that subgroups of finite index are open). Starting with the latter, since J has finite index in W it is clear that its image in $\mathbb{Z}$ has finite index, call it $\pi(J) = m\mathbb{Z}$. But $\pi(\overline{J})$ must contain $m\mathbb{Z}$, moreover, π is an open map so $\pi(\overline{J})$ must be open, and therefore closed. Hence it must contain $m\hat{\mathbb{Z}}$, which has finite index in $\hat{\mathbb{Z}}$. This yields the following exact sequences

$$\begin{tikzcd}
\ker\pi \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[r] & \hat{\mathbb{Z}} \\
\overline{J}\cap\ker\pi \arrow[r, hook] \arrow[u, hook] & \overline{J} \arrow[r, two heads] \arrow[u, hook] & \pi(\overline{J}) \arrow[u, hook]
\end{tikzcd}$$

Since $\overline{J}\cap\ker\pi$ and $\pi(\overline{J})$ are finite index in the top sequence, $\overline{J}$ must be as well.

We now show that $\overline{J}\cap W = J$. We have that $\pi^{-1}(m\hat{\mathbb{Z}}) \subset \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is closed and contains J, so it must contain $\overline{J}$ as well. Since we have that $m\hat{\mathbb{Z}} \subset \pi(\overline{J})$ we can see that $\pi(\overline{J}) = m\hat{\mathbb{Z}}$. Then we have that

$$m\mathbb{Z} = \pi(J) \subset \pi(\overline{J}\cap W) \subset m\hat{\mathbb{Z}}\cap\mathbb{Z} = m\mathbb{Z}$$

which means that $\pi(J) = \pi(\overline{J}\cap W)$. So it will suffice to show that $J\cap\ker\pi =$

$\overline{J} \cap \ker \pi$. To show this, we will consider

$$\overline{J} \cap \ker \pi \big/ J \cap \ker \pi \subset \ker \pi \big/ J \cap \ker \pi.$$

Note that $J \cap \ker \pi$ is open in the profinite $\ker \pi$. So $J \cap \ker \pi$ has finite index in $\ker \pi$, and $\ker \pi \big/ J \cap \ker \pi$ is finite. So it remains to show that $\overline{J} \cap \ker \pi \big/ J \cap \ker \pi$ has no elements of finite order, id est, that it is torsion-free. To see this, begin by recalling that $\pi(J) = m\mathbb{Z}$ and consider the composition

$$\tilde{h} : \mathbb{Z} \cong m\mathbb{Z} \cong J \big/ J \cap \ker \pi \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \Big/ \iota(J \cap \ker \pi).$$

Since $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is profinite we may extend this to a unique

$$h : \hat{\mathbb{Z}} \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \Big/ \iota(J \cap \ker \pi).$$

Now, we have that $h(\mathbb{Z}) = J \big/ J \cap \ker \pi$ and that $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$ so $h(\hat{\mathbb{Z}}) = \overline{J} \big/ J \cap \ker \pi$. Since $\hat{\mathbb{Z}}$ is torsion free it suffices to prove that h is injective. Note that

$$\pi \circ h : \hat{\mathbb{Z}} \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \Big/ \iota(J \cap \ker \pi) \to \hat{\mathbb{Z}}$$

is multiplication by m when restricted to $\mathbb{Z}$. Since $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$ it is true for h without restriction. So $\pi \circ h$ is injective and h is injective. This means that the quotient group $\overline{J} \big/ J \cap \ker \pi$ is torsion free. Consequently, so is $\overline{J} \cap \ker \pi \big/ J \cap \ker \pi$ Finally, we can conclude that $J \cap \ker \pi = \overline{J} \cap \ker \pi$ and that $J = \overline{J} \cap W$. $\square$

Corollary 2.2.12. *The bijection of the previous theorem extends to closed subgroups of H of Gal(K^{ab}/K) such that $\pi(H) = \{0\}$ or $\pi(H)$ has finite index* *in $\hat{\mathbb{Z}}$ and closed subgroups of W.*

Proof. We must emphasis that for the this proof for $X \subset \mathrm{Gal}(K^{\mathrm{ab}}/K)$ the closure $\overline{X}$ is always taken in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and not in an intermediate space. Let J be closed in W. Our first step is to show that $\pi(\overline{J}) = \{0\}$ or that $\pi(\overline{J})$ has finite index in $\hat{\mathbb{Z}}$. Note that $\pi(\overline{J}) = \overline{\pi(J)}$ and that

$$\mathbb{Z} \Big/ \pi(J) \cong \hat{\mathbb{Z}} \Big/ \pi(\overline{J}).$$

Since $\pi(J) = \{0\}$ or $\pi(J) = m\mathbb{Z}$ the result follows.

Let $\iota : W \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the natural embedding that is continuous with respect to each group's topologies from Prop. 2.2.10. We need to show that $\overline{\iota(J)} \cap W = J$.

Case: $\pi(J) = \{0\}$: In this case $J \subset \ker \pi \subset W$. The topology of $\ker \pi$ as a subspace of W coincides with its topology as a subspace of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ by Prop. 2.2.9. Since J is closed in W, it is closed in $\ker \pi$ and $\iota(J)$ is closed in the Galois group.

Case: $\pi(J) = m\mathbb{Z}$: Let $\mathcal{J} = \overline{\iota(J)}$ and note that $\mathcal{J} = \bigcap U$ where U runs through open sets in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ containing $\mathcal{J}$, or equivalently where U is containing $\iota(J)$. Then

$$\mathcal{J} \cap W = \bigcap (U \cap W).$$

But each open $(U \cap W)$ has finite index in the profinite W, so Theorem 2.2.11 allows us to write

$$\mathcal{J} \cap W = \bigcap V$$

where V runs through open sets in W that have finite index, and contain J.

Recall that we have an exact sequence

$$1 \to \ker \pi \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \hat{\mathbb{Z}} \to 1.$$

By also recalling that W is dense in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and that $\pi(W) = \mathbb{Z}$ we also have an exact sequence

$$1 \to \ker \pi \to W \to \mathbb{Z} \to 1,$$

So we may write $W \cong \ker \pi \times \mathbb{Z}$. In particular we can write that $J = C \times m\mathbb{Z}$ for C closed in $\ker \pi$. We write that $C = \cap D$ for D an open set in $\ker \pi_a b$ containing C and then

$$J = \bigcap D \times m\mathbb{Z}.$$

Once again we use that open in a profinite space means of finite index, so that we have

$$\{D \times m\mathbb{Z}\} \subset \{V : \text{open in } W, \text{ of finite index, and } \ J \subset V\}.$$

Hence we have that $\cap V = \cap D \times m\mathbb{Z}$. The other inclusion is clear,

Now let H be a closed subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. We also need to show that $\overline{H \cap W} = H$.

Case: $\pi(H) = \{0\}$ We have that $H \subset \ker \pi \subset W$, so $H \cap W = H$ and is closed in W.

Case: $\pi(H) = m\hat{\mathbb{Z}}$ Let $h \in H$. We will show that $h \in \overline{H \cap W}$. Note that $\mathbb{Z} \cap \pi(H)$ is dense in $\pi(H)$, so we may chose a sequence $h_i \in H$ such that $\pi(h_i) \in \mathbb{Z}$ and $\lim_i \pi(h_i) = \pi(h)$. Since H is closed in a compact space, it is compact, and so we may extract a convergent subsequence h_j from $\{h_i\}$, say $\lim_j h_j = \tilde{h}$. We have that $\pi(\tilde{h}) = \lim_j \pi(h_j) = \pi(h)$. Define $x : \tilde{h}^{-1}h$, so that $x \in \ker \pi \subset W$ and we may see that $x \in H \cap W$ and finally that $h = \lim_j x h_j$. $\square$

These results have some important consequences for the class field theory of global function fields, namely:

Corollary 2.2.13. *$\vartheta_K(\mathbb{A}^*(K)/K^*) = W$, in particular $\vartheta_K : \mathbb{A}^*(K)/K^* \to W$ is an isomorphism of topological groups. Further, W is dense in Gal(K^{ab}/K).*

Proof. That $\vartheta_K(\mathbb{A}^*(K)/K^*) = W$ and that W is dense in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ are well known and found in [1] or [5]. We show that we have an isomorphism of topological groups. Consider $\ker(\pi \circ \vartheta_K) \subset {\mathbb{A}^*(K)}/{K^*}$. We already know that ϑ_K is a continuous bijection. When restricted to $\ker(\pi \circ \vartheta_K)$ we have a continuous bijection of compact groups, hence a homeomorphism. Therefore ϑ_K is an isomorphism of topological groups. $\square$

While Theorem 2.2.5 gave a correspondence between open subgroups of finite index of $\mathbb{A}^*(K)$ and finite abelian extensions of K, we seek a correspondence for the infinite abelian extension K as well. To that end we have that:

Corollary 2.2.14. *Every closed subgroup of $C \subset A^*(K)/K^*$ arises from an abelian extension L/K such that the constant field extension of $\mathbb{F}_q$ is either finite or equal to $\overline{\mathbb{F}_q}$ as the kernel of $\vartheta_{L/K} : \mathbb{A}^*/K^* \to$ Gal$(K^{ab}/K) \to$ Gal(L/K)*

Proof. This is a direct consequence of Corollary 2.2.12 together with the previous corollary; if C is closed in $A^*(K)/K^*$, then $J := \vartheta_K(C)$ is closed in W. Then $H := \overline{J}$ has that $\pi_{\mathrm{ab}}(H) = \{0\}$ or that $\pi_{\mathrm{ab}}(H)$ has finite index in $\hat{\mathbb{Z}}$.

This corresponds to the case of the constant field extension being equal to $\overline{\mathbb{F}_q}$ or finite, respectively. $\square$

2.2.2 Characters and L-functions.

Before we begin, let us state a technical result from [18]. Let $G := \mathrm{Gal}(K/L)$ be the Galois group of a Galois extension of global fields and recall the result of Prop. 2.2.4. Let $I_{\mathfrak{p}}(G)$ denote the inertia subgroup of G, id est, the subgroup that consists of all elements of the decomposition group $G_{\mathfrak{p}}$ that act trivially on the residue field of $\mathfrak{p}$. Then we have:

Theorem 2.2.15. *(Prop. 13, Chapter 13 of [18].) Let L/K be an abelian extension of local fields with Galois group G. Then the local Artin map $\vartheta_{L/K} : K_{\mathfrak{p}} \to G$ sends $\mathcal{O}_{\mathfrak{p}}^*$ onto $I_{\mathfrak{p}}(G)$.*

Let $K^{\mathrm{ab}}(S)$ be an infinite abelian extension of K. The choice of the notation $K^{\mathrm{ab}}(S)$ will become clear in Chapter 3. For a character $\chi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ let

$$U(\chi) := \{\mathfrak{p} \in \mathcal{S}_K \setminus S : \chi|_{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)} = 1\}.$$

Let $\mathcal{D}_S^+(U(\chi))$ be the sub-monoid of effective divisors generated by the primes in $U(\chi)$. For $D \in \mathcal{D}_S^+(U(\chi))$ we set

$$\chi(D) = \chi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)$$

which is well-defined, as difference choices of splits $s_K : \mathcal{D}_S^+(K) \to \mathbb{A}_S^*(K)$ are equivalent up to an element of $\hat{\mathcal{O}}_S^*$.

For any such character χ we know that $\ker \chi$ is an open and closed subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ and as such its fixed field is a finite Galois extension of K, which we denote K_χ. Recall that

$$\mathrm{Gal}(K_\chi/K) \cong \mathrm{Im}\chi \cong \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \big/ \ker\chi .$$

Since $\mathrm{Gal}(K_\chi/K)$ is a finite (and hence closed) subgroup of $\mathbb{T}$ it must be a subgroup of a group of roots of unity, in particular a cyclic group. Conversely, if K' is a finite cyclic extension of K we have that

$$\mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \mathrm{Gal}(K^{\mathrm{ab}}(S)/K') \cong \mathrm{Gal}(K'/K) \hookrightarrow \mathbb{T}$$

allowing us to construct a character χ whose kernel is $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K')$ and image is $\mathrm{Gal}(K'/K)$ so that $K' = K_\chi$.

Lemma 2.2.16. *Let χ be a character of Gal$(K^{\mathrm{ab}}(S)/K)$. The primes in $U(\chi)$ are exactly the primes that are unramified in K_χ.*

Proof. The character $\chi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ induces the following isomorphisms:

$$\mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \big/ \ker\chi \cong \mathrm{Im}\chi \cong \mathrm{Gal}(K_\chi/K).$$

We define an injective character $\bar{\chi} : \mathrm{Gal}(K_\chi/K) \to \mathbb{T}$ by $\bar{\chi}(a + \ker\chi) = \chi(a)$, this map is independent of the choice of representative a. By Theorem 2.2.15 we have that $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) = I_{\mathfrak{p}}(K_\chi/K)$, the inertia subgroup of $\mathrm{Gal}(K_\chi/K)$, hence $\chi \circ \vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) = \bar{\chi}(I_{\mathfrak{p}}(K_\chi/K))$. By Prop 9.6 of [27] we have that the inertia subgroup is trivial, and hence $\chi|_{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)} = 1$, when $\mathfrak{p}$ is unramified in K_χ. □

For any closed subgroup $H \in \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$, let K^H denote the extension of

K that is fixed by H. For any prime $\mathfrak{p} \in \mathcal{S}_K \setminus S$, define

$$N_{\mathfrak{p}} := \bigcap_{\chi : \mathfrak{p} \in U(\chi)} \ker \chi.$$

Lemma 2.2.17. *We have that* $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) = N_{\mathfrak{p}}$ *and that* $K^{N_{\mathfrak{p}}}$ *is the maximal extension of* K *in* $K^{\mathrm{ab}}(S)$ *that is unramified at* $\mathfrak{p}$*, which we denote by* $K^{\mathrm{ab}}(S)^{ur,\mathfrak{p}}$.

Proof. We may discuss the extension $K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)}$ because it is clear that the subgroup $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) \subset \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ is closed: $\vartheta_{K^{\mathrm{ab}}(S)}$ is continuous, $\mathcal{O}_{\mathfrak{p}}^*$ is compact, and Galois groups are Hausdorff.

We again use Theorem 2.2.15, along with the local-global Artin map correspondence from Remark 2.2.6 to show that $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)$ is mapped to the inertial subgroup $I_{\mathfrak{p}}(K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)}/K)$ under the quotient $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathrm{Gal}(K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)}/K)$. This quotient is by the subgroup $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)})$ which is equal to $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)$ by Galois theory. Therefore the inertial group is trivial and $\mathfrak{p}$ is unramified in $K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)}$. In other words, $K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)} \subseteq K^{\mathrm{ab}}(S)^{\mathrm{ur},\mathfrak{p}}$.

Since $N_{\mathfrak{p}}$ is an intersection of a collection of closed sets it is closed, and has an associated field $K^{N_{\mathfrak{p}}}$; moreover,

$$K^{N_{\mathfrak{p}}} = \prod_{\chi : \mathfrak{p} \in U(\chi)} K_{\chi}.$$

Now $K^{\mathrm{ab}}(S)^{\mathrm{ur},\mathfrak{p}}$ must be a composite of finite abelian extensions of K unramified at $\mathfrak{p}$ But any finite abelian extension is the composite of finite cyclic ones, and these are associated to a χ such that $\mathfrak{p} \in U(\chi)$ by the previous lemma, which gives us that

$$K^{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*)} \subseteq K^{\mathrm{ab}}(S)^{\mathrm{ur},\mathfrak{p}} \subseteq K^{N_{\mathfrak{p}}}.$$

But it is also clear that $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) \subseteq N_{\mathfrak{p}}$ because for any $\mathfrak{p} \in U(\chi)$ we have that $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*) \subseteq \ker \chi$. The result then follows. $\square$

We attach to each character χ the **Dirichlet L-function** given by:

$$L_{K,\chi}(z) := \prod_{\mathfrak{p} \in \mathcal{K} \setminus S} (1 - \chi(\mathfrak{p}) N_S(\mathfrak{p})^{-z})^{-1}.$$

By $\chi(\mathfrak{p})$ we mean that $\chi(\mathfrak{p})$ is as previously defined for $\mathfrak{p} \in U(\chi)$ and $\chi(\mathfrak{p}) = 0$ otherwise. This is equivalent to the "usual" L-function for the ring $\mathcal{O}_S(K)$, and because of this we can write it also as a summation:

$$L_{K,\chi}(z) = \sum_{D \in \mathcal{D}_S^+(K)} \chi \circ s_K(D) N_S(D)^{-z}$$

For $\mathfrak{p} \in U(\chi)$ we have from the previous section that if $\pi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathrm{Gal}(K_\chi/K)$ then

$$\pi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p}) = \mathrm{Frob}_{K_\chi/K}(\mathfrak{p}).$$

This allows us to define the L-function as

$$L_{K,\chi}(z) = \prod_{\mathfrak{p} \in U(\chi)} \big(1 - \chi\big(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p})\big) N_S(\mathfrak{p})^{-z}\big)^{-1}.$$

We will always write L-function with both the field and the character as an index (exempli gratia $L_{L,\chi}$), so as to avoid confusion with our notation for field extension of the field L fixed by a character χ (exempli gratia L_χ/L).

2.3 C^*-Algebras

Having concluded our exposition of topics from class field theory that are necessary for later chapters, we turn now to C^* algebras. In Section 2.3.1 we will define a group crossed product and discuss its representations, including induction of representations. This discussion is largely inspired by Chapter VII of [14] and Chapters 2 and 5 of [36]. Section 2.3.2 will cover the primitive ideal space and will include other results from [36] and [32]. We will close our discussion of preliminaries in Section 2.3.3 by studying $\mathbb{R}$-equivariant Morita equivalences. More details on induction of representations and Morita equivalences can be found in [28].

For this section we will let G be a discrete abelian group. We define its Pontryagin dual $\hat{G}$ to be the space $\mathrm{Hom}(G, \mathbb{T})$ of continuous group homomorphism $\chi : G \to \mathbb{T}$. We endow $\hat{G}$ with the compact-open topology, id est, the topology generated by sets $V(H, U) := \{f \in \hat{G} : f(H) \subset U\}$ for H compact in G and U open in $\mathbb{T}$.

Whenever we discuss a representation $\phi : A \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra A we will always mean a representation on a Hilbert space $\mathcal{H}$.

2.3.1 Group Crossed Products & Their Representations

Suppose G acts continuously on a locally compact space X. If $g \in G$ and $x \in X$ we denote this action by $g \cdot x$. There is an automorphism of the C^*-algebra $C_0(X)$ by

$$l : G \to \mathrm{Aut}(C_0(X)) \text{ by } g \mapsto l_g \text{ where } l_g(f)(x) = f(g^{-1} \cdot x).$$

Let $C_c(G, C_0(X))$ denote the algebra of continuous, compactly supported functions on G with values in $C_0(X)$, id est, by formal finite sums $\sum_{g \in G} f_g u_g$. Here u_g denotes the unitary element associated to g. Multiplication is determined by the rule $u_g \cdot f \cdot u_g^{-1} = l_g(f)$, while adjoint is determined by the rule $u_g^* = u_{g^{-1}}$. There is a norm on this algebra by

$$\|\sum_{g \in G} f_g u_g\| = \sup_{\sigma} \|\sigma(\sum_{g \in G} f_g u_g)\|$$

where σ runs through all $*$-representations of the algebra of functions.

Definition 2.3.1. *A group crossed product $C_0(X) \rtimes G$ is the C^*-algebra resulting from completing $C_c(G, C_0(X))$ with respect to the described norm.*

Definition 2.3.2. *A* ***covariant representation*** *of $C_0(X) \rtimes G$ on a Hilbert space $\mathcal{H}$ is a pair (π, U) where $\pi : C_0(X) \to \mathcal{B}(\mathcal{H})$ is a representation of $C_0(X)$ and $U : G \to U(\mathcal{H}))$ is a unitary group representation such that for all $f \in C_0(X)$ and $g \in G$*

$$\pi(l_g(f)) = U_g \pi(f) U_g^*$$

Given a covariant representation (π, U) we can construct a representation $\pi \rtimes U : C_0(X) \rtimes G \to \mathcal{B}(\mathcal{H})$ of a group crossed product on a Hilbert space $\mathcal{H}$ by

$$\pi \rtimes U(\sum_{g \in G} f_g u_g) = \sum_{g \in G} \pi(f_g) U_g$$

and extending by continuity. It can be shown that all representations of $C_0(X) \rtimes G$ arise in this way.

Let H be a subgroup of G. We wish to study how a representation $\phi : C_0(X) \rtimes H \to \mathcal{B}(\mathcal{H})$ can be used to "induce" a new representation $\mathrm{Ind}_H^G \phi$ of $C_0(X) \rtimes G$.

Let us first discuss this in a more general setting. To do that it will be helpful to have the following notion from Chapter 2 of [28]:

Definition 2.3.3. *Let A by a C^*-algebra. By a right (respectively, left)* ***Hilbert A-module*** *E we mean that E is a complex linear space equipped with a right (respectively, left) A-module structure and a map $\langle \cdot, \cdot \rangle_E : E \times E \to A$ that, for $x, y, z \in E$, $a \in A$, and $\alpha, \beta \in \mathbb{C}$ has the following properties:*

- *linearity in its second argument (respectively, first), id est, for $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$ we have that $\langle x, \alpha y + \beta z \rangle_E = \alpha \langle x, y \rangle_E + \beta \langle x, z \rangle_E$;*
- *$\langle x, ya \rangle_E = \langle x, y \rangle_E a$ (respectively, $\langle ax, y \rangle = a \langle x, y \rangle$),*
- *$\langle x, y \rangle_E^* = \langle y, x \rangle_E$,*
- *$\langle x, x \rangle_E \geq 0$,*
- *$\langle x, x \rangle_E = 0 \implies x = 0$.*

We further require that E be complete with respect to the norm $\|x\|_E = |\langle x, x \rangle_E|^{\frac{1}{2}}$, the latter norm being in A. Further, E is said to be full if $\langle E, E \rangle$ is dense in A.

Let A and B be C^*-algebras and let $\phi : B \to \mathcal{B}(\mathcal{H}_\phi)$ be a representation of B. To induce a representation on A will we require a right Hilbert B-module E and a left action of A on E, id est, a map $A \to \mathcal{B}(E)$. We will denote the action of an $a \in A$ on an $e \in E$ by $a \cdot e$. Our new representation $\mathrm{Ind}_E \phi$ of A will be on the the Hilbert space $E \otimes_B \mathcal{H}_\phi$ with inner product

$$\langle e_1 \otimes \xi_1, e_2 \otimes \xi_2 \rangle_{\mathrm{Ind}} := \langle \xi_1, \phi(\langle e_1, e_2 \rangle_E) \xi_2 \rangle_{\mathcal{H}_\phi}$$

Where $\langle\ ,\ \rangle_E$ denotes the inner product of E with values in B and $\langle\ ,\ \rangle_{\mathcal{H}_\phi}$ denotes the inner product of $\mathcal{H}$. Given such a Hilbert module we can define a new representation of A can given by

$$\mathrm{Ind}_E\phi(a)(e \otimes \xi) := a \cdot e \otimes \xi$$

Now let us return to the case of group crossed products. If H is a subgroup of G, $A = C_0(X) \rtimes G$, and $B = C_0(X) \rtimes H$ then we have a natural choice of right Hilbert B-module E, namely **Green's imprimitivity bimodule**. This module is the completion of $C_c(G, C_0(X))$ with respect to the inner product with values in $B = C_0(X) \rtimes H$:

$$\langle f_1 u_s, f_2 u_t \rangle_E = \begin{cases} l_s(f_1 f_2) u_{s^{-1}t} & s^{-1}t \in H \\ 0 & \text{otherwise} \end{cases}$$

Where $f_i \in C_0(X)$ and $s, t \in G$. The action of A on E is via:

$$(f_1 u_s, f_2 u_t) \mapsto f_1 \cdot l_s(f_2) u_{st} \in C_c(G, C_0(X)).$$

With the above we can define an induced representation from B onto A:

Definition 2.3.4. *Let A and B be as defined above with $\phi : B \to \mathcal{B}(\mathcal{H}_\phi)$ a representation of B on a Hilbert space $\mathcal{H}_\phi$. Then $Ind_H^G\phi : A \to \mathcal{B}(A \otimes_B \mathcal{H}_\phi)$ is the induced representation on A via Green's imprimitivity bimodule.*

2.3.2 Primitive Ideals

In this section we will describe the Primitive Ideal space for a special case of the class of C^*-algebras studied in Chapter 8 of [36].

Definition 2.3.5. *Let I be an ideal of a C^* algebra A. I is called* ***primitive*** *if there exists a non-zero irreducible representation $\phi : A \to \mathcal{B}(\mathcal{H})$ such that $I = \ker(\phi)$. $\mathrm{Prim}(A)$ denotes the space of all primitive ideals of A endowed with the hull-kernel topology, id est, the topology whose open sets are of the form $\{P \in \mathrm{Prim}(A) : I \not\subset P\}$ for closed ideals I of A. We typically associated a primitive ideal with its irreducible representation.*

We continue to work with a group crossed product $A = C_0(X) \rtimes G$. A theorem of [36] allows us to describe its primitive ideal space of A with a quotient of $X \times \hat{G}$. Let $G_x = \{g \in G : g \cdot x = x\}$ denote the isotropy group with respect to $x \in X$ and let A_x denote the crossed product C^*-algebras $C_0(X) \rtimes G_x$.

Proposition 2.3.6. *Let $x \in X$ and $\gamma \in \hat{G}$. The pair $(\mathrm{ev}_x, \gamma|_{G_x})$ is a covariant representation of A_x on $\mathbb{C}$.*

Proof. Let $z \in \mathbb{C}$. Since $g \in G_x$ we have $\mathrm{ev}_x(l_g(f))(z) = f(x)z$. On the other hand, we have

$$\gamma(g)\mathrm{ev}_x(f)\gamma(g)^*(z) = \gamma(g)f(x)\overline{\gamma(g)}z = f(x)z$$

$\square$

We define an equivalence relation on $X \times \hat{G}$

$$(x, \gamma) \sim (y, \sigma) \text{ if } \overline{Gx} = \overline{Gy} \text{ and } \gamma\sigma^{-1} \in G_x^{\perp}$$

where $G_x^{\perp} = \{\gamma \in \hat{G} : \gamma(g) = 0 \text{ for all } g \in G_x\}$. By Remark 8.40 of [36] we know that the quotient map to from $X \times \hat{G} \to {X \times \hat{G}}/_{\sim}$ is an open map.

Theorem 2.3.7. *(William's theorem.) The map* $\Phi : (X \times \hat{G}/\sim) \mapsto \mathrm{Prim}(A)$ *given by:*

$$[x, \gamma] \to \ker(Ind_{G_x}^{G}(\mathrm{ev}_x \rtimes \gamma|_{G_x}))$$

is a homeomorphism.

Proof. See Theorem 8.3 in [36]. □

Therefore let us study the representation $\pi = \mathrm{ev}_x \rtimes \gamma|_{G_x}$. Let $\mathcal{H}_{x,\gamma}$ be the Hilbert space of $C_c(G) \otimes_{C_c(G_x)} \mathbb{C}$ completed with respect to the inner product:

$$\langle \xi_s, \xi_t \rangle = \begin{cases} \gamma(s^{-1}t) & \text{if } s^{-1}t \in G_x \\ 0 & \text{otherwise} \end{cases}$$

where $\xi_s : G \to \mathbb{C}$ is the function for with $\xi_s(s) = 1$ and $\xi_s(g) = 0$ for all $g \in G$, $g \neq s$. We then have a representation $\pi_{x,\gamma}$ of A on $\mathcal{H}_{x,\gamma}$ given by

$$\pi_{x,\gamma}(f)(\xi_s) = f(sx)\xi_s,\ \pi_{x,\gamma}(u_t)\xi_s = \xi_{ts}$$

Proposition 2.3.8. *The representation* $\pi_{x,\gamma} : A \to \mathcal{B}(\mathcal{H}_{x,\gamma})$ *is a covariant representation.*

Proof. Using the relations above we see that $\pi_{x,\gamma}(l_g(f))(\xi_s) = f(g^{-1}sx)\xi_s$. On the other hand, $\pi_{x,\gamma}(g)^*(\xi_s) = \xi_{g^{-1}s}$ and so $\pi_{x,\gamma}(f)\pi_{x,\gamma}(g)^*(\xi_s) = f(g^{-1}sx)\xi_{g^{-1}s}$ and finally

$$\pi_{x,\gamma}(g)\pi_{x,\gamma}(f)\pi_{x,\gamma}(g)^*(\xi_s) = f(g^{-1}sx)\xi_s$$

as desired. $\square$

Proposition 2.3.9. *$Ind_{G_x}^G \pi$ is unitary-equivalent to $\pi_{x,\gamma}$*

Proof. This is a special case of Prop. 8.24 in [36], but we work out the details for this specific case. Let $\mathcal{H}_1 = C_c(G, C_0(X)) \otimes_{C_0(X) \rtimes H} \mathbb{C}$ and $\mathcal{H}_2 = C_c(G) \otimes_{C_c(G_x)} \mathbb{C}$, along with their inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ then by our earlier discussion, $\mathrm{Ind}_{G_x}^G \pi : C_0(X) \rtimes G \to \mathcal{B}(\mathcal{H}_1)$ is the representation with the following action:

$$\mathrm{Ind}_{G_x}^G \pi(fu_g)(\eta u_h \otimes z) = fl_g(\eta)u_{gh} \otimes z$$

Note that $f, \eta \in C_0(X)$; $u_g \in C_c(G, C_0(X))$ or, by an abuse of notation, $C_0(X) \rtimes G$; $\xi_g \in C_c(G)$; and $z \in \mathbb{C}$. We first define a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$U(\eta u_h \otimes 1) = \eta(h \cdot x)\xi_h \otimes 1.$$

This means that

$$U(\mathrm{Ind}_{G_x}^G \pi(fu_g))(\eta u_h \otimes z) = f(gh \cdot x)\eta(h \cdot x)\xi_{gh} \otimes z$$

We also note that

$$\pi_{x,\gamma}(fu_g)U(\eta u_h \otimes z) = f(gh \cdot x)\eta(h \cdot x)\xi_{gh} \otimes z \tag{2.3.1}$$

This gives us $U\left(\mathrm{Ind}_{G_x}^{G}\pi\right)U^* = \pi_{x,\gamma}$. We show that U is unitary:

$$\langle f_1 u_s \otimes z_1, f_2 u_t \otimes z_2\rangle_1 =$$

$$z_1\pi(\langle f_1 u_s, f_2 u_t\rangle_E)(z_2) = \begin{cases} \overline{z_1}\overline{f_1(s\cdot x)}f_2(s\cdot x)\gamma(s^{-1}t)z_2 & \text{if } s^{-1}t \in G_x \\ 0 & \text{otherwise} \end{cases} \tag{2.3.2}$$

Whereas

$$\langle U(f_1 u_s \otimes z_1), U(f_2 u_t \otimes z_2)\rangle_2 = \langle f_1(s\cdot x)\xi_s \otimes z_1, f_2(t\cdot x)\xi_t \otimes z_2\rangle_1 =$$

$$\begin{cases} \overline{z_1}\overline{f_1(s\cdot x)}f_2(s\cdot x)\gamma(s^{-1}t)z_2 & \text{if } s^{-1}t \in G_x \\ 0 & \text{otherwise} \end{cases} \tag{2.3.3}$$

□

With this equivalence in hand, we can describe the dimension of the irreducible representations:

Proposition 2.3.10. *(Lemma 2.8 from [32]) For $(x,\gamma) \in X \times \hat{G}$ let $\pi_{x,\gamma} : A \to \mathcal{B}(\mathcal{H}_{x,\gamma})$ be the aforementioned representation. Then $\dim \mathcal{H}_{x,\gamma} = [G : G_x]$. In particular, $\pi_{x,\gamma}$ is finite dimensional if and only if G_x has finite index in G.*

Proof. If $\{s_i\}$ a complete representative set for G/G_x then $\{\xi_{s_i}\}$ is an orthonormal set, which we can see by computing the inner product

$$< \xi_{s_i}, \xi_{s_j} >= \begin{cases} 1 & \text{if } s_i = s_j \\ 0 & \text{otherwise} \end{cases}$$

To see that the span of this set is dense in $\mathcal{H}_{x,\gamma}$ note that the tensor product in $C_c(G) \otimes_{C_c(G_x)} \mathbb{C}$ means that for $s \in G_x$, $t \in G$, and $z \in \mathbb{C}$ we have

$$(\xi_s \xi_t) \otimes z = \xi_t \otimes \gamma(s) z$$

so that representatives of G/G_x are sufficient to span all basis elements. $\square$

2.3.3 Morita Equivalence

This discussion of Morita equivalences is derived from Chapters 2 and 3 of [28].

Definition 2.3.11. *Let A and B be two C^*-algebras. An $A{-}B$* ***imprimitivity bimodule*** *is a set E which is a full Hilbert left A-module and a full Hilbert right B-module (in particular, it has two inner products, ${}_A\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_B$ with values in A, B, respectively) such that ${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$ for all $x, y, z \in E$. If there exists an $A - B$ imprimitivity bimodule, we say that A and B are* ***Morita equivalent****.*

The imprimitivity bimodule allows us to induce ideals and representations from A to B and vice versa in what's called the Rieffel correspondence. Therefore Morita equivalence means that we have an equivalence of the primitive ideal space, in particular:

Theorem 2.3.12. *Let E be an $A - B$ imprimitivity bimodule and π is a representation of B. Then $Ind_E(\pi)$ is irreducible if and only if π is irreducible. Moreover, Induction of representations, when restricted to irreducible representations, gives a homeomorphism of the primitive ideal space of A and B.*

Proof. See Chapter 3 of [28]. $\square$

There is a particular type of relationship between two C^*-algebras - when one is a sub-algebra of another - that allows us to show Morita equivalence.

Definition 2.3.13. *Let A be a C^*-algebra $p \in A$ be a projection. The sub-algebra $B := pAp$ is called a* ***corner****. It is a* ***full corner*** *if, additionally, ApA is dense in A, in which case p is called a* ***full projection****.*

Using the above notation, let $E = pA$. Then E is an $A - B$ imprimitivity bi-module with A-valued inner product ${}_A\langle x, y\rangle = xy^*$ and B-valued inner product $\langle x, y\rangle_B = x^*y$.

Lemma 2.3.14. *(Lemma 2.10 from [32]) Let A be a C^*-algebra and $e \in A$ a full projection so that $E = eA$ is the natural (eAe, A)-imprimitivity bimodule. Let π by a non-degenerate representation of A. Then $Ind_E\pi$ is unitarily equivalent to $(\pi|_{eAe}, \pi(e)\mathcal{H})$. In particular $\dim(Ind_E\pi) = \dim(\pi(e)\mathcal{H})$ when the latter is finite.*

Proof. Let $U : eA \otimes_A \mathcal{H}_\pi \to \pi(e)\mathcal{H}_\pi$ be given by

$$ea \otimes \xi \mapsto \pi(ea)\xi$$

We first show that $U \in \mathcal{B}(eA \otimes_A \mathcal{H}_\pi, \pi(e)\mathcal{H}_\pi)$ is unitary. Note that the inner product on E as a right Hilbert A-module with values in A is given by

$$\langle ea, eb\rangle_{E-A} = (ea)^*eb$$

The inner product on the induced Hilbert space is then

$$\begin{aligned}\langle ea \otimes \xi_1, eb \otimes \xi_2 \rangle_{\mathrm{Ind}} &= \langle \xi_1, \pi(< ea, eb >_{E-A})\xi_2 \rangle_{\mathcal{H}_\pi} \\ &= \langle \xi_1, \pi(ea)^* \pi(eb)\xi_2 \rangle_{\mathcal{H}_\pi} \\ &= \langle \pi(ea)\xi_1, \pi(eb)\xi_2 \rangle_{\mathcal{H}_\pi}\end{aligned}$$

On the other hand, in $\mathcal{H}_\pi$ we have

$$\langle U(ea \otimes \xi_1), U(eb \otimes \xi_2) \rangle_{\mathcal{H}_\pi} = \langle \pi(ea)\xi_1, \pi(eb)\xi_2 \rangle_{\mathcal{H}_\pi}$$

To show unitary equivalence we proceed to examine $U(\mathrm{Ind}_e \pi)$ and $\pi|_{eAe}U$. We have

$$U(\mathrm{Ind}_E)(\pi)(ebe)(ea \otimes \xi) = U(ebea \otimes \xi) = \pi(ebea)\xi$$

On the other hand we also have

$$\pi(ebe)U(ea \otimes \xi) = \pi(ebe)\pi(ea)\xi = \pi(ebea)\xi$$

This shows that $U(\mathrm{Ind}_E)(\pi)(ebe)U^* = \pi(ebe)$ and gives the unitary equivalence of the representations $\mathrm{Ind}_E \pi$ and $\pi|_{eAe}$. □

If we are working with a C^*-dynamical system, by which we mean a pair (A, σ_t) where A is a C^*-algebra and σ_t is a group of automorphisms of A parametrised by $t \in \mathbb{R}$, then we will require an $\mathbb{R}$-equivariant form of the above. Let (A, σ_t^A) and (B, σ_t^B) be two such systems and let E be an (A, B)-imprimitivity bimodule. We follow [32] in the following definition and proposition:

Definition 2.3.15. *The imprimitivity bimodule E is an $\mathbb{R}$-equivariant imprimitivity bimodule if there is a group of isometries U_t on E parametrised by*

$t \in \mathbb{R}$ such that

$$_A\langle U_t x, U_t y\rangle = \sigma_t^A({}_A\langle x, y\rangle)$$

and

$$\langle U_t x, U_t y\rangle_B = \sigma_t^B(\langle x, y\rangle_B)$$

for any $x, y \in E$ and $t \in \mathbb{R}$. If there exists such a bimodule between to dynamical systems then they are said to be $\mathbb{R}$-equivariantly Morita equivalent.

Proposition 2.3.16. *Let E be an $\mathbb{R}$-equivariant imprimitivity bimodule for C^*-dynamical systems (A, σ_t^A) and (B, σ_t^B). Then the Rieffel homeomorphism $Ind_E : PrimB \to PrimA$ is $\mathbb{R}$-equivariant, id est, for a representation $\pi : B \to \mathcal{H}$ the two representations of A are unitary equivalent:*

$$Ind_E(\pi \circ \sigma_t^B) : A \to \mathcal{B}(E \otimes_{\pi\circ\sigma_t^B} \mathcal{H})$$

$$(Ind_E \pi) \circ \sigma_t^A : A \to \mathcal{B}(E \otimes_\pi \mathcal{H})$$

Proof. Let $\mathcal{H}_1$ be the Hilbert space $E \otimes_{\pi\circ\sigma_t^B} \mathcal{H}$ with the inner product

$$\langle e_1 \otimes \xi_1, e_2 \otimes \xi_2\rangle_1 = \langle \xi_1, \pi \circ \sigma_t^B(\langle e_1, e_2\rangle_{E-B})\xi_2\rangle_{\mathcal{H}}$$

and $\mathcal{H}_2$ be the Hilbert space $E \otimes_\pi \mathcal{H}$ with inner product

$$\langle e_1 \otimes \xi_1, e_2 \otimes \xi_2\rangle_2 = \langle \xi_1, \pi(\langle e_1, e_2\rangle_E)\xi_2\rangle_{\mathcal{H}}$$

so that $\mathrm{Ind}_E(\pi \circ \sigma_t^B) : A \to \mathcal{B}(\mathcal{H}_1)$ and $(\mathrm{Ind}_E \pi) \circ \sigma_t^A : A \to \mathcal{B}(\mathcal{H}_2)$. Since E is $\mathbb{R}$-equivariant we have a parameterized group of isometries U_t on E, so let

$T : \mathcal{H}_1 \to \mathcal{H}_2$ by given by

$$T(e \otimes \xi) = U_t(e) \otimes \xi$$

noting that the tensor products are over different rings. We can see that this map is unitary:

$$\begin{aligned}
\langle T(e_1 \otimes \xi_1), T(e_2 \otimes \xi_2)\rangle_2 &= \langle U_t(e_1) \otimes \xi_1, U_t(e_2) \otimes \xi_2\rangle_2 \\
&= \langle \xi_1, \pi(\langle U_t e_1, U_t e_2\rangle_E)\xi_2\rangle_{\mathcal{H}} \\
&= \langle \xi_1, \pi \circ \sigma_t^B(\langle e_1, e_2\rangle_E)\xi_2\rangle_{\mathcal{H}} \\
&= \langle e_1 \otimes \xi_1, e_2 \otimes \xi_2\rangle_1
\end{aligned}$$

Furthermore, we have that

$$\big(T \cdot \mathrm{Ind}_E(\pi \circ \sigma_t^B)(a)\big)(e \otimes \xi) = U_t(ae) \otimes \xi_1$$

While on the other hand we have

$$\big((\mathrm{Ind}_E \pi) \circ \sigma_t^A(a) \cdot T\big)(e \otimes \xi) = \sigma_t^A(a) U_t(e) \otimes \xi = U_t(ae) \otimes \xi$$

We are using the fact that $\sigma_t^A(a)U_t(e) = U_t(ae)$, which follows from the axioms of an $\mathbb{R}$-equivariant imprimitivity bimodule. $\square$

Chapter 3

Constructions involving the ring of S-integers

This chapter is concerned with the construction of Bost-Connes systems and the study of its various ingredients. Our construction of a Bost-Connes system for function fields is largely inspired by [30], which associates a system to an arbitrary extension K'/K and a choice of a finite set S of primes of K to exclude. We interpret these excluded primes as taking the place of the Archimedean places in a number field. Our construction differs in that we only allow an arbitrary choice of S. Once the primes are chosen they will construct for us a field extension $K^{\mathrm{ab}}(S)$. In Section 3.1 we will define the topological monoid which will serve as the underlying topological space for the Bost-Connes system. We will also prove various technical results needed in subsequent chapters. In Section 3.2 we will describe the C^* algebra of the Bost-Connes system as a semigroup crossed product and show how it relates to a group crossed product algebra. Finally in 3.3 we describe how our construction relates to other Bost-Connes systems for function fields.

3.1 The Topological Monoid underlying the Bost-Connes System

Let K be a global function field and let $S \subset \mathcal{S}_K$ be a finite set of primes of K. The goal for this section is to define the underlying monoid for a Bost-Connes systems and study some of its properties. In the number field case the key ingredient is the ring of *finite* adeles and the maximal abelian Galois group. In our function field setting we have two options: we can say that all primes (and thus all adeles) are finite, which is the option taken in [13] and [26]; or we can arbitrarily nominate a finite number of primes to be excluded in the same way that one would exclude the Archimedean primes in the number field case. The former approach lacks an analogue of the ring of integers and analogous objects with the number field case are not well-behaved, in particular its class numbers are not finite. The latter approach is taken in [22] and [30], among others. The downside here is that one is forced to work with a Galois group $\mathrm{Gal}(L/K)$ that is a quotient of the maximal abelian Galois group. However, this approach more closely matches the explicit class field theory described in [21] and [20], where it is shown that

$$K^{\mathrm{ab}} = K^{\mathrm{ab},\mathfrak{p}_1} \cdot K^{\mathrm{ab},\mathfrak{p}_2}$$

where $K^{\mathrm{ab},\mathfrak{p}}$ is the maximal abelian extension of K that is totally split at $\mathfrak{p}$ and where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct. Our approach is similar to [22] in that our choice of prime specifies a field extension, but we allow an arbitrary finite set of primes rather than a single one.

Our construction can be regarded as a special case of [30]. There the author required both a choice of a finite set of primes of K as well as a field extension L/K. In our case the choice of primes enforces a field extension that exhibits certain analogues with the number field case that we will study in the subsequent chapters.

Our construction will proceed as follows: Given S, we will describe a subgroup $B \subset \mathbb{A}^*(K)$. We will embed this subgroup into the Weil group and thus the maximal abelain Galois group $G(K^{\mathrm{ab}}/K)$. Then we will take the fixed field and appropriate quotient. Having assembled the two main ingredients, we will define the underlying monoid for a Bost-Connes system associated to function fields. Along the way we will prove several facts that will be useful in later chapters.

For this section K will always denote a global function field. The map i : $\mathbb{A}_R^*(K) \to A^*(K)$ will always be given by

$$(a_{\mathfrak{p}})_{\mathfrak{p}\notin R} \mapsto (a'_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{S}_L} \quad a'_{\mathfrak{p}} = \begin{cases} 1 & \mathfrak{p} \in R \\ a_{\mathfrak{p}} & \mathfrak{p} \notin R. \end{cases}$$

By abuse of notation we will use i for any set $R \subset \mathcal{S}_K$. We will understand $\mathbb{A}_S^*(K)$ to be implicitly embedded into $\mathbb{A}^*(K)$ via i and will omit writing $i(\mathbb{A}_S^*(K))$ or $i(a_{\mathfrak{p}})$ for $(a_{\mathfrak{p}}) \in \mathbb{A}_S^*(K)$ in that case.

3.1.1 The Subgroup $B \subset \mathbb{A}^*(K)$

Let K_S^* be the diagonal embedding K^* into $\mathbb{A}_S^*(K)$, and then define

$$B := \overline{K_S^* \cdot K^*}$$

where the closure is taken in $\mathbb{A}^*(K)$.

Proposition 3.1.1. *The pre-image $i^{-1}(B)$ in $\mathbb{A}_S^*$ is equal to $\overline{K_S^*}$, moreover, $i^{-1}(B) \subseteq K_S^* \cdot \hat{\mathcal{O}}_S^*$.*

Proof. Let K^{S*} denote the image of K^* under the injection $K^* \hookrightarrow \prod_{\mathfrak{p} \in S} K_{\mathfrak{p}}^*$. Then we have that $K^* \cdot K_S^* = K^{S*} \cdot K_S^*$. From this it follows that $i^{-1}(K^* \cdot K_S^*) = K_S^*$. Note that $i(\mathbb{A}_S^*)$ is closed in $\mathbb{A}_S^*$; to see this it is enough to consider that a converging sequence $\{(a_{\mathfrak{p}}^n)\}_{n=0}^{\infty}$ in $i(\mathbb{A}_S^*)$ must converge to an $(a_{\mathfrak{p}})$ with $a_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \in S$. Because of this fact, and since $i : \mathbb{A}_S^* \to \mathbb{A}^*$ is a homeomorphism onto $i(\mathbb{A}_S^*)$, we have that i preserves closures when restricted to $\mathbb{A}_S^*$, so that $i^{-1}(B) = \overline{K_S^*}$. For the second part note that $K_S^* \cdot \hat{\mathcal{O}}_S^*$ is a union of open sets and is open. Since open subgroups are also closed and since it contains K_S^* we have that $\overline{K_S^*} \subseteq K_S^* \cdot \hat{\mathcal{O}}_S^*$. $\square$

Proposition 3.1.2. *The subgroup B is closed and not open in $\mathbb{A}^*(K)$.*

Proof. Since B is obviously closed we only show that it is not open. Let B_S be the image of B under the projection $\pi : \mathbb{A}^* \twoheadrightarrow \mathbb{A}_S^*$. If B is open then B_S would also be open. We show that this is a contradiction: If B_S were open it would necessarily contain a basic open set in its interior. Following our notation in

Chapter 2, a basic open set U in $\mathbb{A}^*$ is given by

$$U = \prod_{\mathfrak{p}\in R} U_{\mathfrak{p}} \times \prod_{\mathfrak{p}\notin R} \mathcal{O}_{\mathfrak{p}}^*$$

where $U_{\mathfrak{p}}$ is an open subgroup of $K_{\mathfrak{p}}^*$ and where R is a finite subset of $\mathcal{S}_K$, and let $U_S := \pi(U)$. If $U_S \subset B_S$ then by Prop. 3.1.1 we know that $K_S^* \cap U_S$ is dense in U_S. Note that the kernel of the map $x \mapsto (v_{\mathfrak{p}}(x))_{\mathfrak{p}\in R\setminus S}$ for $x \in K_S^* \cap U_S$ is the constant field, $\mathbb{F}_q^*$. This means we have an injective map

$$\mathbb{K}_S^* \cap U_S \Big/ \mathbb{F}_q^* \to \prod_{\mathfrak{p}\in R\setminus S} \mathbb{Z} \times \{0\}.$$

This injection means that $K_S^* \cap U_S$ is finitely generated.

Recall that $1+\mathfrak{p}$ is a direct summand of $\mathcal{O}_{\mathfrak{p}}^*$ and that by [27] Chapter II, Prop. 5.7 we have that $1+\mathfrak{p} \cong \mathbb{Z}_p^N$ (p here refers to the characteristic of the residue field of $\mathcal{O}_{\mathfrak{p}}$.) This means that we may project $1+\mathfrak{p}$ onto $\mathbb{Z}_p$ and that the map

$$U_S \twoheadrightarrow \prod_{\mathfrak{p}\notin R\cup S} \mathcal{O}_{\mathfrak{p}}^* \twoheadrightarrow \prod_{\mathfrak{p}\notin R\cup S} \mathbb{Z}_p \twoheadrightarrow \prod_{\mathfrak{p}\notin R\cup S} \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \prod_{\mathfrak{p}\in F} \mathbb{Z}/p\mathbb{Z}$$

is surjective, for any $F \subset \mathcal{S}_K \setminus R$. But this is a contradiction as U_S is the closure of the finitely generated set $K_S^* \cap U_S$, whereas the minimal number of generators of $\prod_{\mathfrak{p}\in F} \mathbb{Z}/p\mathbb{Z}$ can be arbitrarily large. $\square$

Definition 3.1.3. *By Corollary 2.2.14 we know that B must be associated with an infinite extension. Denote this extension by $K^{\mathrm{ab}}(S)$. It is the field fixed by the subgroup $\vartheta_K(\overline{K_S^* \cdot K^*})$. Denote the corresponding Artin map by $\vartheta_{K^{\mathrm{ab}}(S)} : \mathbb{A}^* \to Gal(K^{\mathrm{ab}}(S)/K)$.*

Note that $\vartheta_{K^{\mathrm{ab}}(S)} = q \circ \vartheta_K$ where $q : \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ is the quotient map.

The next two technical lemmas provide us with important facts for the later chapters.

Lemma 3.1.4. *The image $\vartheta_{K^{\mathrm{ab}}(S)}(\mathbb{A}_S^*)$ is dense in Gal$(K^{\mathrm{ab}}(S)/K)$.*

Proof. By Lemma 2.2.13 we know that $\vartheta_K(\mathbb{A}^*(K))$ is dense in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, therefore it is also dense in

$$\mathrm{Gal}(K^{\mathrm{ab}}(S)/K) = \mathrm{Gal}(K^{\mathrm{ab}}/K) \Big/ \vartheta_K(B).$$

Additionally, we have that

$$\mathbb{A}^*(K) \subset \mathbb{A}_S^*(K) \cdot i(\prod_{\mathfrak{p}\in S} K_{\mathfrak{p}}^*).$$

We will show that $\vartheta_{K^{\mathrm{ab}}(S)}(\mathbb{A}_S^*(K)) = \vartheta_{K^{\mathrm{ab}}(S)}(\mathbb{A}^*(K))$. This amounts to showing that $i(\prod_{\mathfrak{p}\in S} K_p^*) \subset B$. By the Strong Approximation Theorem (2.2.1) K is dense in $\prod_{\mathfrak{p}\in S} K_p$ so for any $(b_{\mathfrak{p}})_{\mathfrak{p}\in S} \in \prod_{\mathfrak{p}\in S} K_p^*$ we can find a sequence k^n such that $(k_{\mathfrak{p}}^n)_{\mathfrak{p}\in S}$ converges to $(b_{\mathfrak{p}})_{\mathfrak{p}\in S}$. Then $(k_{\mathfrak{p}}^n) \cdot i((k_{\mathfrak{p}}^n)_{\mathfrak{p}\notin S})^{-1} = i((k_{\mathfrak{p}}^n)_{p\in S})$ so that $(k_{\mathfrak{p}}^n) \in K^* \cdot K_S^*$. Id est, $i(b_{\mathfrak{p}})_{\mathfrak{p}\notin S} \in B$. $\square$

Lemma 3.1.5. *Let $\mathcal{R} \subset \mathcal{S}_K \setminus S$ be a finite set of primes and for each $\mathfrak{p} \in \mathcal{R}$ let $\xi_{\mathfrak{p}} : K_{\mathfrak{p}}^* \to \mathbb{T}$ be a character on $K_{\mathfrak{p}}^*$. There exists a character χ : Gal$(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ such that $\chi \circ \vartheta_{K^{\mathrm{ab}}(S)}|_{K_{\mathfrak{p}}^*} = \xi_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathcal{R}$.*

Proof. By Theorem 5 of Chapter X of [1] we have that there exists a character $\xi : \mathbb{A}^*(K) \big/ K^* \to \mathbb{T}$ whose restrictions to each $K_{\mathfrak{p}}^*$ for each $\mathfrak{p} \in \mathcal{R}$ is equal to $\xi_{\mathfrak{p}}$ and whose restriction to $K_{\mathfrak{p}}^*$ for each $\mathfrak{p} \in S$ is trivial. We claim that $\xi(B := \overline{K^* \cdot K_S^*}) = 1$. To see this note that continuity of characters means that it suffices to show that $\xi(K^* \cdot K_S^*) = 1$ and that since $K^* \subset \ker \xi$ by definition, we only need that $\xi(K_S^*) = 1$. Any $(a_\mathfrak{p}) \in K_S^*$ can be written as $(a_\mathfrak{p}) = x \cdot (b_\mathfrak{p})$ with $x \in K^*$ and $b_\mathfrak{p} = 1$ for $\mathfrak{p} \notin S$ and $b_\mathfrak{p} = x^{-1}$ for $\mathfrak{p} \in S$. By construction $\xi \circ i(b_\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathcal{S}_K$, so we conclude that $\xi(B) = 1$. The claim means that map extends to a character $\xi : \mathbb{A}^*(K)/B \to \mathbb{T}$. By Prop. 2.2.7 we have that $\vartheta_K : \mathbb{A}^*(K)/B \cong W^{ab}/\vartheta_K(B)$. This proves that there exists a character of the Weil group $\chi := \xi \circ \vartheta^{-1}_{K^{\mathrm{ab}}(S)}$ that exhibits the required properties. The kernels of characters of either the Weil group or the Galois group are in one-to-one correspondence with the open subgroups of finite index, and by Theorem 2.2.11 these subgroups of the Weil group and Galois group are in one-to-one correspondence with each other. In particular, the map from Weil group to the Galois group is given by taking the closure of the subgroup in question. $\square$

Proposition 3.1.6. *For $\mathfrak{p}' \in \mathcal{S}_K \setminus S$, the map $\iota_{\mathfrak{p}'} : K^*_{\mathfrak{p}'} \to \mathbb{A}^*(K)/B$ by $x \mapsto [(a_\mathfrak{p})]$, where $a_{\mathfrak{p}'} = x$ and $a_\mathfrak{p} = 1$ for all other $\mathfrak{p} \neq \mathfrak{p}'$, is injective.*

Proof. Let $x \in \ker \iota_{\mathfrak{p}'}$, $x \neq 1$ and choose a character $\xi \in \hat{K^*_{\mathfrak{p}'}}$ such that $\xi(x) \neq 1$. By Lemma 3.1.5 we have a character $\chi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ such that $\chi \circ \vartheta_K|_{K^*_{\mathfrak{p}'}} = \xi$ and $\chi \circ \vartheta_K|_{K^*_\mathfrak{p}} = 1$ for all $\mathfrak{p} \in S$. We have a commutative triangle

$$\begin{tikzcd}
K^*_{\mathfrak{p}'} \arrow[r, "\iota_{\mathfrak{p}'}"] \arrow[dr, "\vartheta_{K^{\mathrm{ab}}(S)} \circ i"'] & \mathbb{A}^*(K)/B \arrow[d, "\vartheta_{K^{\mathrm{ab}}(S)}"] \\
& \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)
\end{tikzcd}$$

Therefore $1 = \chi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ \iota_{\mathfrak{p}'}(x) = \chi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ i(x) = \xi(x)$, which is a contradiction. $\square$

This next lemma uses notation from Section 2.2.2.

Proposition 3.1.7. *Let $\mathfrak{p}' \in \mathcal{S}_K \backslash S$ with $N := N_S(\mathfrak{p}')$. There exists a character $\chi : Gal(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ such that $\chi(\mathfrak{p}') = e^{\frac{2\pi i}{k}}$ for some $k \in \mathbb{N}$, $k > 2$ and $\chi(\mathfrak{p}) = 1$ for all other $\mathfrak{p} \in \mathcal{S}_K \setminus S$ with $N_S(\mathfrak{p}) = N$. We say that such a character is a **character with distinguished prime** $\mathfrak{p}'$.*

Proof. The proof of Prop. 2.1.2 tells us that there are only finitely many primes $\mathfrak{p}$ with $N_S(\mathfrak{p}) = N$. For each prime $\mathfrak{p} \neq \mathfrak{p}'$ such that $N_S(\mathfrak{p}) = N$ let $\xi_{\mathfrak{p}} : K_{\mathfrak{p}}^* \to \mathbb{T}$ be the local trivial character, and for $\mathfrak{p}'$ let $\xi'_{\mathfrak{p}} : K_{\mathfrak{p}'}^* \to \mathbb{T}$ be a character such that $\xi_{\mathfrak{p}'}(\pi_{\mathfrak{p}'}) = e^{\frac{2\pi i}{k}}$ for some $k \geq 3$ and $\xi_{\mathfrak{p}'}(\mathcal{O}_{\mathfrak{p}}^*) = 1$. The result follows from Lemma 3.1.5. □

3.1.2 An Example

Before continuing on to define the underlying monoid of our Bost-Connes system we will explore an explicit example of a $K^{\mathrm{ab}}(S)$. While function fields do not generally have a distinguished prime, the rational function field $k = \mathbb{F}_q(T)$ does; let ∞ denote the prime whose valuation is

$$v_\infty\left(\frac{f(T)}{g(T)}\right) = \deg(g) - \deg(f).$$

We will use the Carlitz module to construct a class of extensions of k that have a distinguished finite subset S of primes lying above ∞.

Let $k\langle\tau\rangle$ be the ring of polynomials in τ with coefficients in $k = \mathbb{F}_q(T)$. The multiplication in $k\langle\tau\rangle$ is : $\tau a = a^q \tau$ for all $a \in k$. The Carlitz module is defined as the $\mathbb{F}_q$-algebra homomorphism

$$\rho : \mathbb{F}_q[T] \to k\langle\tau\rangle \text{ given by } T \mapsto \rho_T = T + \tau.$$

For example, we can compute ρ_{T^2}:

$$\rho_{T^2} = \rho_T \rho_T = (T+\tau)(T+\tau) = \tau^2 + (T^q + T)\tau + T^2.$$

For any non-zero polynomial $a \in F_q[T]$ we note that $\rho_a(\tau)$ is a polynomial with coefficients in k. So we can ask for solutions to the equation $\rho_a(\tau) = 0$. This leads us to define the $F_q[T]$ module:

$$\Lambda_\rho(a) := \{\lambda \in \bar{k} : \rho_a(\lambda) = 0\}$$

It is known (in say, [29] and others) that the field $K_{\rho,a} := k(\Lambda_\rho(a))$ is an abelian extension of k. This class of extensions does not produce all abelian extensions. For one, these fields are geometric and their union does not include the maximal extension of the constant field. In [21] Hayes proved an analogue of the Kronecker-Weber theorem for global function fields - every abelian extension of k is a compositum of the union of all such $K_{\rho,a}$, $\bar{\mathbb{F}}_q k$, and the union of all fields L_n, where L_n is a specific sub field of $k(\Lambda_{\rho'}(a))$, with $\rho' : \mathbb{F}_q[\frac{1}{T}] \to k\langle\tau\rangle$ being the Carlitz module for $\mathbb{F}_q[\frac{1}{T}]$. In particular we have:

Theorem 3.1.8. *Let $S_{\rho,a}$ denote the set of primes in $K_{\rho,a}$ lying above ∞. Then*

$$\#S_{\rho,a} = \frac{\Phi(a)}{q-1}$$

Where

$$\Phi(a) = \#\left(\mathbb{F}_q[T] \big/ (a)\right).$$

Proof. See Chapter 12 of [29]. $\square$

From the above theorem, we can see fields of the form $K_{\rho,T-c}$ for $c \in \mathbb{F}_q$ have a single distinguished prime; the prime ∞ either ramifies completely or remains inert. These fields are an analogue of the imaginary quadratic number fields.

Proposition 3.1.9. *Let $K = K_{\rho,T-c}$ and $S = S_{\rho,T-c} = \{\infty\}$ as above. The group of units $\mathcal{O}_S^*$ is finite.*

Proof. Combine Theorem 3.1.8 with Prop. 2.1.1. $\square$

We will revisit this example in Section 4.2.1.1 when we consider the primitive ideal space of our Bost-Connes algebra and its relation to the S-class number, which we denote by h_S. By Theorem 2.1.2 we know that h_S is finite for non-empty S. In principle one could compute this number for $S = S_{\rho,T-c}$, or any other S. In practice this is quite difficult. Prop. 2.1.3 suggests one might began by calculating size of the class group of 0-degree divisors. Some examples of computed S-class numbers can be found in [2], though these are for various sets S resulting from computing a ray class field and not necessarily from the example discussed here.

3.1.3 The underlying monoid and actions of the divisors

Consider the topological space $\mathbb{A}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$. This space comes with a natural associative multiplication from the multiplication on the ring $\mathbb{A}_S(K)$ and the group $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ turning it into a topological monoid. $\hat{\mathcal{O}}_S^*(K)$ acts on this monoid by

$$(r_{\mathfrak{p}}) \cdot ((a_{\mathfrak{p}}), \gamma) \mapsto ((a_{\mathfrak{p}} r_{\mathfrak{p}}), \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})^{-1}\gamma)$$

Definition 3.1.10. *The topological monoid $X_{K,S}$ is defined as*

$$X_{K,S} := \mathbb{A}_S(K) \times \mathit{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \hat{\mathcal{O}}_S^*(K).$$

It is easy to see that multiplication of $\mathbb{A}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ is well-defined in the quotient, which $X_{K,S}$ then inherits.

Fix a standard split $s : \mathcal{D}_S(K) \to \mathbb{A}_S^*(K)$ and let $(d_{\mathfrak{p}}) := s(D)$ denote the image of an S-divisor $D = \sum_{\mathfrak{p}} D_{\mathfrak{p}}\mathfrak{p}$.

Definition 3.1.11. *$\mathcal{D}_S(K)$ acts on $X_{K,S}$ by:*

$$D \cdot [(a_{\mathfrak{p}}), \gamma] = [(a_{\mathfrak{p}} d_{\mathfrak{p}}), \vartheta_{K^{\mathrm{ab}}(S)}(d_{\mathfrak{p}})^{-1}\gamma]$$

It is important to note that the quotient by $\hat{\mathcal{O}}_S^*(K)$ means that this action does not depend on the choice of split. It is also clear that $D \cdot ([(a_{\mathfrak{p}}), \gamma] \cdot [(b_{\mathfrak{p}}), \delta]) = (D \cdot [(a_{\mathfrak{p}}), \gamma]) \cdot [(b_{\mathfrak{p}}), \delta]$.

Definition 3.1.12. *The* ***Deligne-Ribet monoid*** *$Y_{K,S}$ is the topological submonoid of $X_{K,S}$:*

$$Y_{K,S} := \hat{\mathcal{O}}_S(K) \times \mathit{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \hat{\mathcal{O}}_S^*(K).$$

$Y_{K,S}$ inherits the multiplication and unit of $X_{K,S}$; it is clear that it is closed under this multiplication.

We will only consider the action of the effective divisors $\mathcal{D}_S^+(K)$ on $Y_{K,S}$, as an arbitrary divisor may take an element of $Y_{K,S}$ to the larger space $X_{K,S}$. Note that $Y_{K,S}$ is both open under the subspace topology and compact.

We have a result regarding this action that is analogous to Lemma 4.1 in [11] concerning the maximal abelian Galois group for global fields:

Lemma 3.1.13. *The image if the map*

$$\iota : \mathcal{D}_S^+(K) \to Y_{K,S} \;\; by \;\; \iota(D) = D \cdot [1,1] = [s(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s(D)^{-1}]$$

is dense in $Y_{K,S}$.

Proof. The proof follows from two steps:

1. The subspace

$$Z = \coprod_{D \in \mathcal{D}_S^+} \{[s(D), \alpha] : \alpha \in \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)\}$$

   is dense in $Y_{K,S}$.

2. The subspace Z is contained in the closure of the image of $\mathcal{D}_S^+$ in $Y_{K,S}$.

For the first step, let $[(a_\mathfrak{p}), \alpha] \in Y_{K,S}$ and enumerate the primes of $\mathcal{S}_K \setminus S$ as $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots\}$. Since for each prime $\mathfrak{p}$, elements of the local ring $\mathcal{O}_\mathfrak{p}$ can be written as $r \cdot \pi_p^n$ where $r \in \mathcal{O}_p^*$, $n \in \mathbb{N}$, and π_p a uniformizer; we can write $(a_\mathfrak{p}) = (r_\mathfrak{p}) \cdot (u_\mathfrak{p})$ where $(r_\mathfrak{p}) \in \hat{\mathcal{O}}_S^*(K)$ and each $u_\mathfrak{p} = \pi_\mathfrak{p}^{n_p}$, $n_\mathfrak{p} \in \mathbb{N} \cup \{\infty\}$. Define an effective divisor $D_j = \sum_{i=1}^{j} \min(n_p, j)\mathfrak{p}_j$. Then we have $s(D_j)$ converges to $(u_\mathfrak{p})$ as $j \to \infty$ in $\hat{\mathcal{O}}_S(K)$. By definition $[(a_\mathfrak{p}), \alpha] = [(u_\mathfrak{p}), \vartheta_B(r_\mathfrak{p})\alpha]$, so we have also found a sequence $[s(D_j), \vartheta_B(r_\mathfrak{p})\alpha]$ in Z that converges to $[(a_\mathfrak{p}), \alpha]$.

For the second step we need to show that given an $\alpha \in \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ and an effective divisor D, we can find a sequence of effective divisors D_j such that

$$\lim_{j\to\infty}[s(D_j), \vartheta_B \circ s(D_j)^{-1}] = [s(D), \alpha].$$

Let $K = K_0 \subset K_1 \subset K_2 \subset \ldots \subset K^{\mathrm{ab}}(S)$ be a tower of finite extensions of K contained within $K^{\mathrm{ab}}(S)$ so that we have $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K) = \varprojlim \mathrm{Gal}(K_j/K)$ and define $p_j : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathrm{Gal}(K_j/K)$ to be the quotient map. By Chebotarev's Density Theorem ([29] Theorem 9.13A) we know that for each $\sigma \in \mathrm{Gal}(K_j/K)$ that there are infinitely many primes $\mathfrak{p} \in \mathcal{S}_K \setminus S$ unramified in K_j such that $F_{K_j/K}(\mathfrak{p}) = \sigma$, where $F_{K_j/K}$ is the Frobenius automorphism. Additionally, for $\mathfrak{p}$ unramified in K_i we have by Theorem 2.2.5 that $p_j \circ \vartheta_B \circ s(\mathfrak{p}) = F_{K_j/K}(\mathfrak{p})$, which informs us that the map $p_j \circ \vartheta_B \circ s : R \to \mathrm{Gal}(K_i/K)$ is surjective for every co-finite subset $R \subset \mathcal{S}_K \setminus S$. In particular, for each $j \in \mathbb{N}$ we can chose a prime $\mathfrak{p}_j$ such that:

$$p_j(\vartheta_B \circ s(\mathfrak{p}_j)^{-1}) = p_j(\alpha \cdot \vartheta_B \circ s(D))$$

and such that we never chose the same prime twice. Now let $D_j = D + \mathfrak{p}_j$. For each j we have that $\vartheta_B \circ s(D_j)^{-1}$ maps to the same element as α under p_j, thus $\vartheta_B \circ s(D_j)$ converges to α and the result follows. □

3.2 Crossed product C^*-algebras

In this section will we define the Bost-Connes system for a global function field excluding a finite, non-empty set of primes S. We will show that it is Morita equivalent to dynamical system of a group crossed product. The results in this section are analogs of basic facts from other versions of a Bost-Connes system.

Let $1_{Y_{K,S}} \in C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ be the projection which takes values 1 on elements of $Y_{K,S}$ and 0 otherwise.

Definition 3.2.1. *The **Bost-Connes** C^* **algebra** for a global function field K excluding primes S is the C^* algebra $\mathcal{A}_{K,S}$*

$$\mathcal{A}_{K,S} := 1_{Y_{K,S}}\big(C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)\big)1_{Y_{K,S}}.$$

We define a time evolution on this algebra via the S-divisor norm:

$$\sigma_t(f) = f \quad \sigma_t(u_D) = N_S(D)^{it}u_D$$

$(\mathcal{A}_{K,S}, \sigma_t)$ *is the **Bost-Connes system**.*

It is clear from the definition that $\mathcal{A}_{K,S}$ is a corner of group crossed product. To enable us to use the machinery developed in in Section 2.3, our next task will be to show Morita equivalence between $\mathcal{A}_{K,S}$ and $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$. Note that the time evolution from Definition 3.2.1 also applies to $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$.

Proposition 3.2.2. *$\mathcal{A}_{K,S}$ is a full corner of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$. Consequently, $\mathcal{A}_{K,S}$ and $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ are $\mathbb{R}$-equivariantly Morita equivalent.*

Proof. First let us recall the action of $\mathcal{D}_S$ on $X_{K,S}$:

$$D \cdot [(a_\mathfrak{p}), \gamma] = [(a_\mathfrak{p} d_\mathfrak{p}), \vartheta(d_\mathfrak{p})^{-1}\gamma]$$

By considering the orbit $\mathcal{D}_S \cdot \hat{\mathcal{O}}_S$, we can see that

$$\mathbb{A}_S = \bigcup_{D \in \mathcal{D}_S} D \cdot \hat{\mathcal{O}}_S(K)$$

since the action of a divisor $n\mathfrak{p} \cdot \hat{\mathcal{O}}_S(K)$ introduces all possible valuations at the $\mathfrak{p}$-place. Therefore we can write that

$$X_{K,S} = \bigcup_{D \in \mathcal{D}_S} D \cdot Y_{K,S}.$$

Let $E \subset C_0(X_{K,S})$ denote the sub-algebra of functions that have support in only a finite union of $D \cdot Y_{K,S}$. Note that E is:

- Closed under complex conjugation.
- For each $x = [(a_\mathfrak{p}), \gamma] \in X_{K,S}$, there is some element $f \in E$ such that $f(x) = 0$.
- For each pair $x, y \in X_{K,S}$, $x \neq y$ there is an $f \in E$ such that $f(x) \neq f(y)$.

To verify the last claim note that $Y_{K,S}$ is compact and Hausdorff thus $C(Y_{K,S})$ separates points, so it follows immediately if there exists a divisor D such that $x, y \in D \cdot Y_{K,S}$. If that is not the case, and D is a divisor such that $x \in D \cdot Y_{K,S}$ and $y \notin D \cdot Y_{K,S}$, then we can choose any continuous function f with support only in $D \cdot Y_{K,S}$ and with $f(x) \neq 0$; clearly $f(x) \neq f(y)$. These claims combine to give us that E is dense in $C_0(X_{K,S})$ by the Stone-Weierstrass theorem for locally compact spaces.

Let V be the span of elements $\{f 1_{Y_{K,S}} g : f, g \in C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)\}$. Clearly V contains elements of the form $f u_D 1_{Y_{K,S}} g u_{-D}$ with $f, g \in C_0(X_{K,S})$ and

$D \in \mathcal{D}_S(K)$.

$$fu_D 1_{Y_{K,S}} g u_{-D}(x) = f l_D(1_{Y_{K,S}}) l_D(g)(x) = \begin{cases} f(x)g((-D)\cdot x) & \text{if } x \in D \cdot Y_{K,S} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the span of such functions contains the space of continuous functions with support in a finite union of $D \cdot Y_{K,S}$, thus $E \subset V$ and $C_0(X_{K,S}) \subset \overline{V}$. Note that $C_0(X_{K,S})$ embedded as $C_0(X_{K,S})u_0$ in $C_0(X_{K,S}) \rtimes \mathcal{D}_S$ generates the entire space as an ideal (u_0 is the unitary related to the 0-divisor). Hence V is dense in $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$.

We have shown that $\mathcal{A}_{K,S}$ is a full corner of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$. To see $\mathbb{R}$-equivariance note that our $A$$-$$B$ imprimitivity bimodule is $E = 1_{Y_{K,S}}(C_0(X_{K,S} \rtimes \mathcal{D}_S(K))$ with $A = C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ and $B = 1_{Y_{K,S}} A 1_{Y_{K,S}}$. Let $U_t = \sigma_t|_E$. To show that this is a family of isometries first note that the norm on E is given by

$$\|x\|_E = \|{}_A\langle x, x\rangle\| = \|xx^*\| = \|x^*x\| = \|\langle x, x\rangle_B\|$$

for $x \in E$. By linearity it suffices to show that U_t is an isometry for elements of the form $x = 1_{Y_{K,S}} f u_D$. So then we have that

$$\|U_t x\|_E = \|{}_A\langle U_t x, U_t x\rangle\| = \|{}_A\langle N_S(D)^{it} x, N_S(D)^{it} x, \rangle\| = \|{}_A\langle x, x\rangle\|\| = \|x\|_E.$$

It remains to show that the family obeys the laws:

$${}_A\langle U_t x, U_t y\rangle = \sigma_t({}_A\langle x, y\rangle)$$

and

$$\langle U_t x, U_t y\rangle_B = \sigma_t(\langle x, y\rangle_B)$$

for any $x, y \in E$. Again it will suffice to show this for $x = 1_{Y_{K,S}} f u_D$ and $y = 1_{Y_{K,S}} g u_F$. For the first relation we start by computing

$$\begin{aligned}
\sigma_t({}_A\langle x, y\rangle) &= \sigma_t(xy^*) \\
&= \sigma_t(1_{Y_{K,S}} f u_D u_F^* g^* 1_{Y_{K,S}}) \\
&= \sigma_t(1_{Y_{K,S}} f g^* l_{D-F}(1_{Y_{K,S}}) u_{D-F}) \\
&= N_S(D-F)^{it} 1_{Y_{K,S}} f g^* l_{D-F}(1_{Y_{K,S}}) \\
&= N_S(D-F)^{it} xy^*.
\end{aligned}$$

On the other hands we have

$$\begin{aligned}
{}_A\langle U_t x, U_t y\rangle &= {}_A\langle N_S(D)^{it} x, N_S(F)^{it} y\rangle \\
&= N_S(D-F)^{it} xy^* \\
&= \sigma_t(xy^*).
\end{aligned}$$

Similarly, for the inner product with values in B we also have that

$$\sigma_t(\langle x, y\rangle_B) = \sigma_t(x^* y) = N_S(F-D)^{it} x^* y.$$

And we complete the proof by computing:

$$\begin{aligned}
\langle U_t x, U_t y\rangle_B &= \langle N_S(D)^{it} x, N_S(F)^{it} y\rangle_B \\
&= N_S(F-D)^{it} x^* y \\
&= \sigma_t(x^* y).
\end{aligned}$$

$\square$

For clarity's sake we will describe the primitive ideals of $\mathcal{A}_{K,S}$ in terms of the better-understood $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ using the machinery from Section 2.3.3.

Proposition 3.2.3. *Let* $\pi_{x,\gamma} : C_0(X_{K,S}) \rtimes \mathcal{D}_S(K) \to \mathcal{B}(\mathcal{H}))$ *denote the irreducible representation described in Section 2.3.2. Then* $\pi^0 : \mathbb{A}_{K,S} \to \mathcal{B}(\mathcal{H}^0)$ *with* $\pi^0 = \pi_{x,\gamma}|_{\mathcal{A}_{K,S}}$ *and* $\mathcal{H}^0 = \pi_{x,\gamma}(1_{Y_{K,S}})\mathcal{H}_{x,\gamma}$ *is unitarily equivalent to the representation of* $\mathbb{A}_{K,S}$ *that corresponds to* $\pi_{x,\gamma}$ *under the Rieffel homeomorphism.*

Proof. This is a straightforward application of Lemma 2.3.14 with $e = 1_{Y_{K,S}}$. The sub-algebra $1_{Y_{K,S}} \cdot C_0(X_{K,S}) \rtimes \mathcal{D}_S(K) \cdot 1_{Y_{K,S}}$ is equal to $\mathcal{A}_{K,S}$, so the result follows. □

3.3 Relationship with other constructions

It is already clear that our construction of the Bost-Connes system is a special case of that in [30]. The construction in [22] allows for a single choice of prime $\mathfrak{p}_0$, from which the author constructs via Drinfeld modules an extension $\mathbb{K}/K$ that satisfies

$$K^{\mathrm{ab},\mathfrak{p}_0} \subset \mathbb{K} \subset K^{\mathrm{ab}}$$

where by $K^{\mathrm{ab},\mathfrak{p}_0}$ we mean the maximal abelian extension of K that is totally split at $\mathfrak{p}_0$. If we let $S = \{\mathfrak{p}_0\}$ then by Prop. 4.2.8 we have that K_S is closed in $\mathbb{A}_S^*(K)$. Further, the same proof works for empty S, id est, K is also closed in $\mathbb{A}^*(K)$. This means that $B = K_S \cdot K$. We can also write as $B = K \cdot K_{\mathfrak{p}_0}$, where we treat $K_{\mathfrak{p}_0}$ as though it were embedded into $\mathbb{A}^*(K)$ with 1's in the places $\mathfrak{p} \neq \mathfrak{p}_0$. It was shown in [30] (Prop. 1.3.7) that the fixed field of $K \cdot K_{\mathfrak{p}_0}$ is in fact $K^{\mathrm{ab},\mathfrak{p}_0}$.

Chapter 4

Primitive Representations and their dynamics

This chapter investigates the C^*-algebra of the Bost-Connes system along with its dynamics. We show that the methods of Takeishi in [32] can be used to derive new results for analogous objects in the function field case. The first of our two main results is:

Theorem 4.1. *Let K be a global function field and and S a non-empty finite subset of primes of K. If h_S denotes the S-class number of K and $\mathcal{A}_{K,S}$ the C^*-algebra of the Bost-Connes system for K at S then the irreducible representations of $\mathcal{A}_{K,S}$ have either dimension h_S or have infinite dimension.*

The analogous theorem of Takeishi recovers the *narrow* class number of number fields from the primitive ideals of the C^*-algebra. As discussed in Chapter 2, the narrow class group excludes the real Archimedean primes. Our philosophy is to use the finite, non-empty subset S of primes of K to be an arbitrary choice of "Archimedean" primes, and thus the S-class group is the appropriate object in the function field setting. The proof of this result relies on William's Theorem 2.3.7 in the same manner as Takeishi, though the connection to the class field theory reflects the differences in the function field case. The result is obtained entirely from the C^*-algebra of the Bost-Connes system and does not rely on its dynamics.

When considering the dynamics, we are able to proof our second main result:

Theorem 4.2. *Let K and L be function fields such that there exists finite sets of primes S and R of K and L, respectively, which yield $\mathbb{R}$-equivariantly isomorphic Bost-Connes systems $(\mathcal{A}_{K,S}, \sigma_t^K)$ and $(\mathcal{A}_{L,R}, \sigma_t^L)$. Then there exists a group isomorphism $\phi : \mathcal{P}_S(K) \to \mathcal{P}_R(L)$ that preserves the S-norm map, id est, $N_R(\phi(\mathrm{div}_S(f))) = N_S(\mathrm{div}_S(f))$ for all $f \in K$.*

These two results together generalize the work in [32] to the function field setting.

Section 4.1 of this chapter will explore the action of the S-divisor group $\mathcal{D}_S(K)$ on the underlying monoid X_{K_S}. In doing so, we'll develop a proof Theorem 4.1. The main work is done in Prop 4.1.1, which relates the isotropy group to the kernel of a special quotient of the Artin map.

Section 4.2 computes the primitive ideal space of the underlying C^* algebra and investigates the dynamics of it. We show that the methods and results of [32] on the primitive ideal space are fully compatible with the framework described here and, together with the dynamics of the C^* algebra, can be used to recover dual of the principal S-divisor group and S-norm. The section concludes with a proof of Theorem 4.2.

4.1 The action of $\mathcal{D}_S(K)$ on $X_{K,S}$

Let $x \in X_{K,S}$. We know from Chapter 3 that $\mathcal{D}_S(K)$ acts on $X_{K,S}$, so let $\mathcal{D}_{S,x}$ denote the isotropy group of S-divisors that fix x.

Proposition 4.1.1. *Let K be a global function field, S a finite, non-empty subset of $\mathcal{S}_K$, and let $x = [(a_\mathfrak{p}), \gamma] \in X_{K,S}$. If $(a_\mathfrak{p}) = 0$ then $[\mathcal{D}_S(K) : \mathcal{D}_{S,x}] = |\mathcal{C}l_S|$, the S-class number of K, otherwise $[\mathcal{D}_S(K) : \mathcal{D}_{S,x}] = \infty$.*

Proof. Suppose $(a_\mathfrak{p}) \neq 0$. Let $\mathfrak{r}$ be a prime of K with $a_\mathfrak{r} \neq 0$, and let $D = \sum_\mathfrak{p} D_\mathfrak{p}\mathfrak{p} \in \mathcal{D}_{S,x}$. Recall that, assuming we have fixed a standard split s, we write $(d_\mathfrak{p}) = s(D)$. As $D \cdot x = x$, we have that $(a_\mathfrak{p} d_\mathfrak{p}) = (a_\mathfrak{p} r_\mathfrak{p}^{-1})$ for some $(r_\mathfrak{p})^{-1} \in \hat{\mathcal{O}}_S^*(K)$, this means that we have $a_\mathfrak{r} d_\mathfrak{r} r_\mathfrak{r} = a_\mathfrak{r}$, in other words, that $d_\mathfrak{r} \in \mathcal{O}_\mathfrak{r}^*$ and $D_\mathfrak{r} = 0$. Therefore we see that the S-divisor $n\mathfrak{r} - m\mathfrak{r}$ is not in $\mathcal{D}_{S,x}$ when $n \neq m$. That is, $n\mathfrak{r} \notin m\mathfrak{r} + \mathcal{D}_{S,x}$ and so we must have infinitely many distinct cosets in $\mathcal{D}_S(K) \big/ \mathcal{D}_{S,x}$.

Now let $(a_\mathfrak{p}) = 0$. By definition we have that $\mathcal{D}_{S,x} = \{D \in \mathcal{D}_S(K) : D \cdot [0, \gamma] = [0, \gamma]\}$. Recall the action of $\mathcal{D}_S(K)$ on $X_{K,S}$: $D \cdot [(a_\mathfrak{p}), \gamma] = [(a_\mathfrak{p} d_\mathfrak{p}), \vartheta_{K^{\mathrm{ab}}(S)}(d_\mathfrak{p})^{-1}\gamma]$. Since $X_{K,S}$ is defined up to a quotient of the image $\hat{\mathcal{O}}_S^*(K)$ under the Artin map, we have:

$$\mathcal{D}_{S,x} = \{\sum_\mathfrak{p} D_\mathfrak{p}\mathfrak{p} \in \mathcal{D}_S(K) : \vartheta_{K^{\mathrm{ab}}(S)}(d_\mathfrak{p}) = \vartheta_{K^{\mathrm{ab}}(S)}(r_\mathfrak{p}) \text{ for some } (r_\mathfrak{p}) \in \hat{\mathcal{O}}_S^*(K)\}.$$

In other words, $\mathcal{D}_{S,x}$ is the kernel of the map

$$\mathcal{D}_S(K) \xrightarrow{s} \mathbb{A}_S^*\mathbb{A}^* \xrightarrow{\vartheta_{K^{\mathrm{ab}}(S)}} \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \big/ \vartheta_{K^{\mathrm{ab}}(S)}\big(\hat{\mathcal{O}}_S^*(K)\big).$$

Note that quotient by an image of $\hat{\mathcal{O}}_S^*(K)$ on the far right hand side means that the full composition does not depend on the choice of split $s : \mathcal{D}_S(K) \to \mathbb{A}^*_S(K)$.

Consider the kernel of the composition $\mathbb{A}^*_S \xrightarrow{i} \mathbb{A}^* \xrightarrow{\vartheta_{K^{\mathrm{ab}}(S)}} \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$. By definition the kernel of $\vartheta_{K^{\mathrm{ab}}(S)}$ is $\overline{K^* \cdot K^*_S}$, therefore kernel of the composition is equal to $i^{-1}(\overline{K^* \cdot K^*_S}) = \overline{K^*_S}$ by Prop. 3.1.1. By the Fundamental Theorem of Galois theory, together with Definition 3.1.3, we have that

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \Big/ \vartheta_K(\overline{K^* \cdot K^*_S}) \cong \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$$

and therefore we also have

$$\begin{aligned} & \mathrm{Gal}(K^{ab}/K) \Big/ \vartheta_K(\overline{K^* \cdot K^*_S} \cdot \hat{\mathcal{O}}^*_S(K)) \\ \cong\ & \mathrm{Gal}(K^{ab}/K) \Big/ \Big(\vartheta_K(\overline{K^* \cdot K^*_S}) \cdot \vartheta_K(\hat{\mathcal{O}}^*_S(K))\Big) \\ \cong\ & \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \vartheta_{K^{\mathrm{ab}}(S)}\big(\hat{\mathcal{O}}^*_S(K)\big) \end{aligned}$$

again by Definition 3.1.3. It follows that the kernel of the map

$$\mathbb{A}^*_S \to \mathbb{A}^* \to \mathrm{Gal}(K^{ab}/K) \Big/ \vartheta_K(\overline{K^* \cdot K^*_S} \cdot \hat{\mathcal{O}}^*_S(K))$$

is $i^{-1}(\overline{K^* \cdot K^*_S}) \cdot \hat{\mathcal{O}}^*_S(K) = K^*_S \cdot \hat{\mathcal{O}}^*_S(K)$ by Prop. 3.1.1. If we note that

$$s^{-1}(K^*_S \cdot \hat{\mathcal{O}}^*_S(K)) = ad(K^*_S \cdot \hat{\mathcal{O}}^*_S(K)) = \mathcal{P}_S(K),$$

then we have that $\mathcal{D}_{S,x} = \mathcal{P}_S(K)$. □

Proposition 2.3.10 together with the above result means that the irreducible representations of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ have either infinite dimension or dimension equal to the S-class number of K. We have demonstrated earlier that $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ is Morita equivalent to the Bost-Connes algebra $\mathcal{A}_{K,S}$ associated to K with primes S excluded. As stated, Morita equivalence gives us an equivalence of the categories of representations of the two algebras, but this categorical equivalence says nothing about preserving the dimension of the representations themselves. We show this with the following result:

Corollary 4.1.2. *The irreducible representations of the C^* algebra $C_0(X_{K,S}) \rtimes \mathcal{D}_S$ are either of infinite dimension or of dimension equal to the S-class number of K. Consequently, the same is true for $\mathcal{A}_{K,S}$.*

Proof. The first part is obvious, so we only show that the irreducible representations of $\mathcal{A}_{K,S}$ have the same dimension. Let $x = [\rho, \alpha] \in X_{K,S}$ and $\gamma \in \hat{\mathcal{D}}_S$. Let the pair $(\pi_{x,\gamma}, \mathcal{H})$ denote the irreducible representation of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ described in Section 2.3.2. By Prop. 3.2.3 we know that the representation $\pi^0 = \pi_{x,\gamma}|_{\mathcal{A}_{K,S}}$ on $\mathcal{H}^0 = \pi_{x,\gamma}(1_{Y_{K,S}})\mathcal{H}_{x,\gamma})$ is unitary equivalent to the corresponding representations of $\mathcal{A}_{K,S}$. If $\rho = 0$ then

$$\pi_{x,\gamma}(1_{Y_{K,S}})(\xi_D) = 1_{Y_{K,S}}(D \cdot [0, \alpha])\xi_D = \xi_D$$

by definition of $\pi_{x,\gamma}$ and because $D \cdot [0, \alpha] = [0, \alpha \cdot \vartheta_{K^{\mathrm{ab}}(S)} \circ s(D)] \in Y_{K,S}$ for any $D \in \mathcal{D}_S(K)$, and so $\dim \pi^0 = \dim \pi_{x,\gamma}$ by Lemma 2.3.14. This is shown to be the S-class number by Prop 4.1.1. It remains to show that π^0 is infinite dimensional when $\rho \neq 0$.

Choose an effective divisor $D \in \mathcal{D}_S^+$ such that $D \cdot x \in Y_{K,S}$. Let $\mathfrak{p}$ be a prime of K such that $\rho_{\mathfrak{p}} \neq 0$ Then the proof of Proposition 4.1.1 shows us that each $\mathfrak{p}^n$ is distinct in $\mathcal{D}_S/\mathcal{D}_{S,x}$ for $n \in \mathbb{Z}$, and therefore so are $\mathfrak{p}^n + D$. This means that

$\{\xi_{\mathfrak{p}^n+D}\}$ is an orthogonal family in $\mathcal{H}_{x,\gamma}$. Finally, as $(\mathfrak{p}^n+D)x \in Y_{K,S}$, we have that $\xi_{\pi^n+D} \in \pi_{x,\gamma}(1_{Y_{K,S}})\mathcal{H}_{x,\gamma}$ and $\pi_{x,\gamma}(1_{Y_{K,S}})\mathcal{H}_{x,\gamma}$ is infinite dimensional. $\square$

Theorem 4.1 follows immediately.

4.1.1 Technical Lemmas

The following three results will be used in the next section, but are included here because the both topic and methods used are similar. Let S be a non-empty finite set of primes of K. For a subset R of $\mathcal{S}_K \setminus S$ define the symbol Γ_R as:

$$\Gamma_R = \{ad(a_\mathfrak{p}) : a \in \overline{K_S^*} \subset \mathbb{A}_S^*,\ a_\mathfrak{p} = 1 \text{ for } \mathfrak{p} \notin R\} \subset \mathcal{D}_S(K)$$

and for $x = [(a_\mathfrak{p}), \gamma] \in X_{K,S}$, let the symbol R_x be the set $\{\mathfrak{p} \in \mathcal{S}_K : a_\mathfrak{p} = 0\}$.

Remark 4.1.3. *It is clear from the definitions that* $\Gamma_{\mathcal{S}_K \setminus S} = \mathcal{P}_S(K)$.

Lemma 4.1.4. *For* $x \in X_{K,S}$*, the isotropy group* $\mathcal{D}_{S,x} = \Gamma_{R_x}$ *as sets.*

Proof. Recall the action of $\mathcal{D}_S(K)$ on $X_{K,S}$:

$$D \cdot [(a_\mathfrak{p}), \gamma] = [(d_\mathfrak{p} a_\mathfrak{p}), \vartheta_{K^{\mathrm{ab}}(S)}(d_\mathfrak{p})^{-1}\gamma]$$

Where $(d_\mathfrak{p})$ is the image of D under a standard split. If $x = [(a_\mathfrak{p}), \gamma]$ is fixed under this action then we must have that $s(D) \in \hat{\mathcal{O}}_S^*(K) \cdot \ker \vartheta_{K^{\mathrm{ab}}(S)}$ and that $v_\mathfrak{p}(D) = 0$ when $a_\mathfrak{p} \neq 0$, so that $d_\mathfrak{p} = 1$. But applying the definitions of R_x and Γ we have

$$\Gamma_{R_x} = \{ad(b_\mathfrak{p}) : b \in \overline{K_S^*},\ b_\mathfrak{p} = 1 \text{ for } a_\mathfrak{p} \neq 0\}$$

Additionally, since $\overline{K_S^*} = i^{-1}(\overline{K^* \cdot K_S^*})$ by Prop. 3.1.1 and since $\hat{\mathcal{O}}_S^*(K) \in \ker ad$, the condition that $s(D) \in \hat{\mathcal{O}}_S^*(K) \cdot \ker \vartheta_{K^{\mathrm{ab}}(S)}$ is equivalent to $D = ad(b_\mathfrak{p})$ for $(b_\mathfrak{p}) \in \overline{K_S^*}$. $\square$

Lemma 4.1.5. *Let* $[(a_\mathfrak{p}), \gamma] \in X_{K,S}$, *then*

$$\overline{\mathcal{D}_S(K) \cdot [(a_\mathfrak{p}), \gamma]} = \{[(b_\mathfrak{p}), \delta] \in X_{K,S} : a_\mathfrak{p} = 0 \implies b_\mathfrak{p} = 0\}$$

Proof. Let $H = \{[(b_\mathfrak{p}), \delta] \in X_{K,S} : a_\mathfrak{p} = 0 \implies b_\mathfrak{p} = 0\}$. We see that $\mathcal{D}_S(K) \cdot [(a_\mathfrak{p}), \gamma]$ is a subset of H. To show the other inclusion we will find a sequence in the orbit of $[(a_\mathfrak{p}), \gamma]$ that converges to $[(b_\mathfrak{p}), \delta]$. Lemma 3.1.4 allows us to choose a sequence $(c_\mathfrak{p}^n) \in \mathbb{A}_S^*$ such that $\vartheta_{K^{\mathrm{ab}}(S)}(c_\mathfrak{p}^n) \to \gamma\delta^{-1}$. For each $(c_\mathfrak{p}^n)$ define

$$C_n = \sum_{\mathfrak{p} \notin S} v_\mathfrak{p}(c_\mathfrak{p}^n)\mathfrak{p} = ad(c_\mathfrak{p}^n) \in \mathcal{D}_S.$$

Then $C_n \cdot [(a_\mathfrak{p}), \gamma] = [(c_\mathfrak{p}^n a_\mathfrak{p}), \vartheta_{K^{\mathrm{ab}}(S)}(c_\mathfrak{p}^n)^{-1}\gamma]$. The following lemma allows us to choose another sequence $k_m \in K_S^*$ such that, for fixed n, $k_m(c_\mathfrak{p}^n a_\mathfrak{p})$ converges to $(b_\mathfrak{p})$ as $m \to \infty$. Since $\vartheta_{K^{\mathrm{ab}}(S)}(k_m) = 1$, we may take converging sub-sequences and re-index to get a new sequence

$$(\mathrm{div}(k_l) + C_l) \cdot [k_l \cdot (a_\mathfrak{p}), \gamma)]$$

that converges to $[(b_\mathfrak{p}), \delta]$ as $l \to \infty$. $\square$

Lemma 4.1.6. *Let* $(a_\mathfrak{p}) \in \mathbb{A}_S(K)$. *Then* $\overline{K_S^*(a_\mathfrak{p})} = \{(b_\mathfrak{p}) \in \mathbb{A}_S : a_\mathfrak{p} = 0 \implies b_\mathfrak{p} = 0\}$.

Proof. Let $H = \{(b_\mathfrak{p}) \in \mathbb{A}_S(K) : a_\mathfrak{p} = 0 \implies b_\mathfrak{p} = 0\}$. Clearly $K_S^*(a_\mathfrak{p}) \subset H$. We will show that $K_S^*(a_\mathfrak{p})$ is dense in H. Note that $\mathbb{A}_S = K_S \cdot \hat{\mathcal{O}}_S(K)$. So if

$r \in \mathcal{O}_S(K)$, $r \neq 0$, then r is obviously in K^* and so $K_S^* r(a_\mathfrak{p}) = K_S^*(a_\mathfrak{p})$. The set H is also invariant under multiplication by K^* (or indeed any S-idele), so it will suffice to prove the result for $(a_\mathfrak{p}) \in \hat{\mathcal{O}}_S(K)$. Note that we may project $p : H \to \mathbb{A}_{R\cup S}(K)$ where $R = \{\mathfrak{p} \in \mathcal{S} \setminus S : a_\mathfrak{p} = 0\}$. In fact, this projection is a homeomorphism, with an obvious inverse of adding 0 in the $\mathfrak{p} \in R$ places. By the Strong Approximation Theorem 2.2.1 $p(K^*(a_\mathfrak{p}))$ is dense in $\mathbb{A}_{R\cup S}$. □

4.2 The Primitive Ideal Space

William's Theorem 2.3.7 gives us a recipe to study the primitive ideal space of the Bost-Connes algebra, we simply need to calculate the equivalence relation on $X_{K,S} \times \hat{\mathcal{D}}_S(K)$:

$$(x, \gamma) \sim (y, \delta) \text{ if } \overline{\mathcal{D}_S(K) \cdot x} = \overline{\mathcal{D}_S(K) \cdot y} \text{ and } \gamma\delta^{-1} \in \mathcal{D}_{S,x}^{\perp}$$

We will continue to use the notation Γ_R and R_x for a subset R of $\mathcal{S}_K$ and $x \in X_{K,S}$, respectively, as in the previous section. The technical lemmas in the previous section already allow us to prove a major result:

Theorem 4.2.1. *There exists a bijection (as sets) Prim*$(\mathcal{A}_{K,S}) \to \bigsqcup_{R\subset \mathcal{S}_K \setminus S} \hat{\Gamma}_R$.

Proof. Since $\mathcal{A}_{K,S}$ and $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ are Morita equivalent we have a functorial correspondence between their representations which preserves irreducibility. So it suffices to consider the primitive ideal space of the latter algebra.

William's theorem 2.3.7 states that $\mathrm{Prim}(C_0(X_{K,S}) \rtimes \mathcal{D}_S(K))$ is homeomorphic to $X_{K,S} \times \hat{\mathcal{D}}_S(K) /_\sim$ where $([(a_\mathfrak{p}), \gamma], \chi_1) \sim ([(b_\mathfrak{p}), \delta], \chi_2)$ when we have

1. $\overline{\mathcal{D}_S(K)\cdot[(a_\mathfrak{p}),\gamma]} = \overline{\mathcal{D}_S(K)\cdot[(b_\mathfrak{p}),\delta]}$ and

2. $\chi_1\chi_2^{-1} \in \mathcal{D}^{\perp}_{S,[(a_\mathfrak{p}),\gamma]}$

Define a map $\Psi : X_{K,S} \times \hat{\mathcal{D}}_S(K) \to \bigsqcup_{R\subset\mathcal{S}_\mathcal{K}\setminus\mathcal{S}} \hat{\Gamma}_R$ by

$$([(a_\mathfrak{p}),\gamma],\chi_1) \mapsto \Psi(([(a_\mathfrak{p}),\gamma],\chi_1)) = \chi_1|_{\Gamma_{R_{[(a_\mathfrak{p}),\gamma]}}}$$

Assume that $([(a_\mathfrak{p}),\gamma],\chi_1) \sim ([(b_\mathfrak{p}),\delta],\chi_2)$. By Lemma 4.1.5 the first condition means that $(a_\mathfrak{p})$ and $(b_\mathfrak{p})$ are zero at the same primes. This implies that $R_{[(a_\mathfrak{p}),\gamma]} = R_{[(b_\mathfrak{p}),\delta]}$. If we combine the same lemma with Lemma 4.1.4 we have that $\mathcal{D}_{S,[(a_\mathfrak{p}),\gamma]} = \mathcal{D}_{S,[(b_\mathfrak{p}),\delta]}$. The second condition means that χ_1 and χ_2 agree on all divisors in this set. Together we can conclude that

$$\chi_1|_{\Gamma_{R_{[(a_\mathfrak{p}),\gamma]}}} = \chi_2|_{\Gamma_{R_{[(b_\mathfrak{p}),\delta]}}}$$

in other words, this map preserves equivalence classes, so let

$$\tilde{\Psi} : X_{K,S} \times \hat{\mathcal{D}}_S(K)\,/\!\sim\; \to \bigsqcup_{R\subset\mathcal{S}_\mathcal{K}\setminus\mathcal{S}} \hat{\Gamma}_R$$

be the map on the quotient space. To show that it is a bijection we will explicitly define an inverse. For $\chi \in \hat{\Gamma}_R$ define $(a_\mathfrak{p}) \in \mathbb{A}_S$ by

$$a_\mathfrak{p} = \begin{cases} 1 & \mathfrak{p} \notin R \\ 0 & \mathfrak{p} \in R \end{cases}$$

and let $\chi' \in \hat{\mathcal{D}}_S(K)$ be given by

$$\chi'(D) = \begin{cases} \chi(D) & \in \mathcal{D}_{S,[(a_\mathfrak{p}),1]} = \Gamma_R \\ 1 & \text{otherwise} \end{cases}$$

We assert that the map Φ defined by taking χ to the equivalence class represented by $([(a_\mathfrak{p}),1],\chi')$ is an inverse. If $([(b_\mathfrak{p}),\delta],\chi'') \sim ([(a_\mathfrak{p}),1],\chi')$ then we have, as previously discussed, that $R_{[(a_\mathfrak{p}),1]} = R_{[(b_\mathfrak{p}),\delta]}$ and that χ' and χ'' agree on divisors $D \in \mathcal{D}_{S,[(a_\mathfrak{p}),1]}$, so we may conclude that

$$\chi''|_{\Gamma_{R_{[(b_\mathfrak{p}),\delta]}}} = \chi'|_{\Gamma_{R_{[(a_\mathfrak{p}),1]}}} = \chi.$$

In other words, $\Phi \circ \tilde{\Psi}([(b_\mathfrak{p}),\delta],\chi'') = [[(b_\mathfrak{p}),\delta],\chi'']$. By definition we have that $\chi'|_{\Gamma_{R_{[(a_\mathfrak{p}),1]}}} = \chi$, so $\tilde{\Psi} \circ \Phi(\chi) = \chi$ for $\chi \in \hat{\Gamma}_R$. □

Remark 4.2.2. *Before continuing our discussion of the primitive ideal space, let us recall from Remark 4.1.3 that $\mathcal{P}_S(K) = \Gamma_{\mathcal{S}_K \setminus S}$. From the above we can embed $\hat{\mathcal{P}}_S(K) \hookrightarrow \bigsqcup_{R \subset \mathcal{S}_K \setminus S} \hat{\Gamma}_R$. Let us briefly describe the image of this embedding. If $\chi \in \hat{\mathcal{P}}_S(K)$ then the inverse bijection Φ described above would take $\Phi(\chi)$ to the equivalence class represented by $([0,1],\chi')$. The second condition of the equivalence relation on $X_{K,S} \times \hat{\mathcal{D}}_S(K)\,/_{\sim}$ means that each χ gets sent to its own equivalence class. Altogether this means that $\Phi(\hat{\mathcal{P}}_S(K)) = \{[(0,1),\chi'] : \chi \in \hat{\mathcal{P}}_S(K)\}$.*

The dual group $\hat{\Gamma}_R$ has a non-trivial topology, though the above theorem says nothing about the topology of either space. Following Takeishi we will describe the topology of Prim$\mathcal{A}_{K,S}$ via a continuous, open surjective map with a well-understood topological space. But first we need to generalize a result of Laca and Raeburn from [25].

The **quasi-orbit space** $\mathcal{Q}(\mathbb{A}_S/K^*)$ of $\mathbb{A}_S(K)$ is the quotient of the adele ring by the equivalence relation defined by $(a_\mathfrak{p}) \sim (b_\mathfrak{p})$ iff $\overline{K_S^*(a_p)} = \overline{K_S^*(b_p)}$. We will endow it with the quotient topology from $\mathbb{A}_S$. As we have already stated and used, Lemma 4.1.6 shows that $(a_\mathfrak{p}) \sim (b_\mathfrak{p})$ iff they have the same support. The space of supports of an adele lies in the power set $2^{\mathcal{S}_K \setminus S}$, which we equip with the **power-cofinite topology** wherein the topology is generated by a basis consisting of sets of the form:

$$U_R := \{T \in 2^{\mathcal{S}_K \setminus S} : T \cap R = \varnothing\}$$

where R is any *finite* subset of $\mathcal{S}_K \setminus S$.

Proposition 4.2.3. *The map* $\rho : \mathcal{Q}(\mathbb{A}_S/K^*) \to 2^{\mathcal{S}_K \setminus S}$ *given by*

$$[(a_\mathfrak{p})] \mapsto R_{(a_\mathfrak{p})} := \{\mathfrak{p} \in \mathcal{S}_K : a_\mathfrak{p} = 0\}$$

is a homeomorphism if $2^{\mathcal{S}_K \setminus S}$ *is endowed with the power-cofinite topology.*

Proof. By a lemma of [17] we know that the quotient map $q : \mathbb{A}_S \to \mathcal{Q}(\mathbb{A}_S/K^*)$ is open and continuous. From the above discussion we know that we have a bijection between $\mathcal{Q}(\mathbb{A}_S/K^*)$ and $2^{\mathcal{S}_K \setminus S}$. Let $T \in \mathcal{S}_K \setminus S$ be a finite subset of primes excluding S, then we can write a basic open set of $\mathbb{A}_S(K)$ as

$$U := \prod_{\mathfrak{p} \in T} U_\mathfrak{p} \times \prod_{\mathfrak{p} \notin T} \mathcal{O}_\mathfrak{p},$$

where $U_\mathfrak{p} \subset K_\mathfrak{p}$ is an open subgroup. It is clear that $q(U)$ is a basic open set on the quasi-orbit space. We want to show that $\{\rho \circ q(U) : U \text{ open in } \mathbb{A}_S(K)\}$

is the same set as $\{U_R : U_R \text{ open in } 2^{\mathcal{S}_K \setminus S}\}$. Now for $U \subset \mathbb{A}_S(K)$ as above, the set $\{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\} \in 2^{\mathcal{S}_K \setminus S}$ is finite and every finite subset of $\mathcal{S}_K \setminus S$ arises for some U. Therefore we only need to prove that for every open U of $\mathbb{A}_S(K)$ we have that $\rho \circ q(U) = U_{\{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\}}$.

Now for $(a_\mathfrak{p}) \in U$ we have that $a_\mathfrak{p} \in U_\mathfrak{p}$ for all $\mathfrak{p} \in T$. Thus $R_{(a_\mathfrak{p})}$ and $\{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\}$ cannot intersect, so $\rho(a_\mathfrak{p}) \in U_{\{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\}}$.

Now suppose $F \in U_{\{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\}}$, id est, F is a subset of primes not in S such that $F \cap \{\mathfrak{p} \in T : 0 \notin U_\mathfrak{p}\} = \varnothing$. We need to find a $(a_\mathfrak{p}) \in U$ such that $\rho(a_\mathfrak{p}) = F$. If $\mathfrak{p} \in F$ choose $a_\mathfrak{p} = 0$; for $\mathfrak{p} \notin F$ but $\mathfrak{p} \in T$ choose $a_\mathfrak{p} \in U_\mathfrak{p} \setminus \{0\}$; if $\mathfrak{p} \notin F \cup T$ let $a_\mathfrak{p} = 1 \in \mathcal{O}_p$. $\square$

We will need to extend the homeomorphism $\rho : \mathcal{Q}(\mathbb{A}_S/K^*) \to 2^{\mathcal{S}_K \setminus S}$ to a map $X_{K,S} \to 2^{\mathcal{S}_K \setminus S}$. By Lemma we have that 4.1.6 $(r_\mathfrak{p} a_\mathfrak{p}) \in \overline{K_S^*(a_\mathfrak{p})}$ for any $(r_\mathfrak{p}) \in \hat{\mathcal{O}}_S^*(K)$. This means that the map $q : X_{K,S} \to \mathcal{Q}(\mathbb{A}_S/K^*)$ by $[(a_\mathfrak{p}), \gamma] \mapsto [(a_\mathfrak{p})]$ is well-defined so we may define a composition $\rho' : X_{K,S} \to 2^{\mathcal{S}_K \setminus S}$ by $[(a_\mathfrak{p}), \gamma] \mapsto \rho([(a_\mathfrak{p})])$. Note that ρ' is not injective, but it is both open and continuous. That it is open follows immediately from q being open, which is apparent after considering the following commutative diagram where all other maps are quotients or projections, which are open:

$$\begin{array}{ccc}
\mathbb{A}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) & \longrightarrow & \mathbb{A}_S(K) \\
\downarrow & & \downarrow \\
X_{K,S} & \xrightarrow{\quad q \quad} & \mathcal{Q}(\mathbb{A}_S/K^*).
\end{array}$$

This fact is necessary for us to study the topology of the primitive ideal space.

Proposition 4.2.4. *Identifying $X_{K,S} \times \hat{\mathcal{D}}_S(K)\,/_\sim$ with $\bigsqcup_{R \in 2^{\mathcal{S}_K \setminus S}} \hat{\Gamma}_R$, we have* *that the map*

$$Q : 2^{\mathcal{S}_K \setminus S} \times \hat{\mathcal{D}}_S(K) \to X_{K,S} \times \hat{\mathcal{D}}_S(K) /_{\sim} \text{ by } Q(R, \chi) = \chi|_{\Gamma_R} \in \hat{\Gamma}_R$$

is an open, continuous surjection.

Proof. Let $Q' : X_{K,S} \times \hat{\mathcal{D}}_S(K) \to X_{K,S} \times \hat{\mathcal{D}}_S(K) /_{\sim}$ be the quotient map. By [36] its an open continuous map. From the above discussion we have that $Q' = Q \circ (\rho', Id)$. Since Q' and (ρ', Id) are open, continuous surjective maps Q must also be. □

4.2.1 Recovering $\mathcal{P}_S(K)$

Remark 4.1.3 means that $\hat{\mathcal{P}}_S(K)$ is embedded as a distinguished subspace into Prim$\mathcal{A}_{K,S}$ and the last proposition tells us that its homeomorphic onto its image, since $\hat{\mathcal{P}}_S = Q(\mathcal{S}_K \setminus S, \hat{\mathcal{D}}_S)$. This allows us to extract another number-theoretic invariant, but we extract it from the full C^*-dynamical system rather than just the algebra:

Proposition 4.2.5. *If K and L are global function fields such that their exists finite sets of primes S and R of K and L, respectively, with $\mathbb{R}$-equivariantly isomorphic Bost-Connes systems $(\mathcal{A}_{K,S}, \sigma_t^K) \cong (\mathcal{A}_{L,R}, \sigma_t^L)$ then $\hat{\mathcal{P}}_S(K)$ and $\hat{\mathcal{P}}_R(L)$ are $\mathbb{R}$-equivariantly homeomorphic.*

Before proceeding with the proof note that we implicitly defined an $\mathbb{R}$-action on Prim$\mathcal{A}_{K,S}$ in Prop. 2.3.16 by $t \cdot \ker \pi = \ker(\pi \circ \sigma_t)$. We will need to describe this action in terms of the quotient space of $X_{K,S} \times \hat{\mathcal{D}}_S(K)$:

Proposition 4.2.6. *The aforementioned $\mathbb{R}$-action on Prim$(C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)) \cong$*

$X_{K,S} \times \hat{\mathcal{D}}_S(K)\,/_{\sim}$ *is given by*

$$\sigma_t([a,\chi]) = [a, N_S(\cdot)^{it}\chi]$$

for $a \in X_{K,S}$, $\chi \in \hat{\mathcal{D}}_S(K)$ *and* N_S *the* S*-divisor norm on* K*.*

Proof. For $[a,\chi] \in X_{K,S} \times \hat{\mathcal{D}}_S(K)\,/_{\sim}$ we use the notation and corresponding results from Section 2.3. That is, the associated irreducible representation $\pi_{a,\chi}$ is on the Hilbert space $\mathcal{H}_{a,\chi}$, which is the completion of $C_c(\mathcal{D}_S) \otimes_{C_c(\mathcal{D}_{S,a})} \mathbb{C}$ with respect to the inner product

$$\langle \xi_F, \xi_D \rangle = \begin{cases} \chi(F-D) & \text{if } F - D \in \mathcal{D}_{S,a} \\ 0 & \text{otherwise} \end{cases}$$

where $\xi_D \in \mathcal{H}_{a,\chi}$ is the indicator function for the divisor D. Additionally, u_D is the unitary element in $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ associated to $D \in \mathcal{D}_S(K)$ and $f \in C_0(X_{K,S})$. Define $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ so that $[a,\chi] \mapsto \big(\pi_1, \mathcal{H}_1\big)$ and $[a, N_S(\cdot)^{it}\chi] \mapsto \big(\pi_2, \mathcal{H}_2\big)$, respectively, under the map given in William's Theorem 2.3.7. We need to show unitary equivalence between the two operators $\pi_1 \circ \sigma_t(\cdot)$ and $\pi_2(\cdot)$. Define the operator

$$U_t : \mathcal{H}_1 \to \mathcal{H}_2 \text{ by } \xi_D^1 \mapsto N_S(D)^{-it}\xi_D^2$$

We show that this is unitary:

$$\begin{aligned} \langle U_t\xi_D^1, U_t\xi_F^1 \rangle_2 \;&=\; N_S(D)^{-it}\overline{N_S(F)^{-it}}\langle \xi_D^2, \xi_F^2 \rangle_2 \;= \\ & N_S(F-D)^{-it}\langle \xi_D^2, \xi_F^2 \rangle_2 \;=\; \langle \xi_D^1, \xi_F^1 \rangle_1 \end{aligned} \tag{4.2.1}$$

We also note that

$$U_t\big(\pi_1 \circ \sigma_t(u_F)\big)(\xi_D^1) = N_S(F)^{it} U_t \xi_{D+F}^1 = N_S(D)^{-it} \xi_{D+F}^2$$

and

$$U_t(\pi_i \circ \sigma_t(f))\xi_D^1 = f(D \cdot a) U_t \xi_D^1 = N_S(D)^{-it} f(D \cdot a) \xi_D^2$$

while on the other hand, we have that

$$\pi_2(u_F) U_t(\xi_D^1) N_S(D)^{-it} \pi_2(u_F) \xi_D^2 = N_S(D)^{-it} \xi_{D+F}^2$$

and

$$\pi_2(f) U_t(\xi_D^1) = N_S(D)^{-it} \pi_2(f) \xi_D^2 = N_S(D)^{-it} f(D \cdot a) \xi_D^2.$$

This means that $U_t(\pi_1 \circ \sigma_t) U_t^* = \pi_2$, so we have unitary equivalence between $\pi_1 \circ \sigma_t$ and π_2 as desired. □

Proof. Of Prop. 4.2.5.

Recall from Remark 4.2.2 that $\hat{\mathcal{P}}_S(K) \hookrightarrow \bigsqcup_{R \subset \mathcal{S}_K \setminus S} \hat{\Gamma}_R$ and by Theorem 4.2.1 that $\hat{\mathcal{P}}_S(K)$ is a subspace of the primitive ideal space of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$. In particular, $\chi \in \hat{\mathcal{P}}_S(K)$ is mapped to an equivalence class represented by $([(a_\mathfrak{p}), 1], \chi')$ where $(a_\mathfrak{p}) = 0$. Therefore by the proof of Prop. 4.1.1 we have that $\hat{\mathcal{P}}_S(K)$ corresponds to irreducible representations of $C_0(X_{K,S}) \rtimes \mathcal{D}_S(K)$ of finite dimension. By Lemma 2.3.14 we know that the elements of $\mathrm{Prim}\mathcal{A}_{K,S}$ that correspond to $\Phi(\hat{\Gamma}_{\mathcal{S}_K \setminus S})$ under the $\mathbb{R}$-equivariant homeomorphism of Prop. 2.3.16 are also of finite dimension.

If $(\mathcal{A}_{K,S_K}, \sigma_t^K)$ and $(\mathcal{A}_{L,S_L}, \sigma_t^L)$ are $\mathbb{R}$-equivariantly isomorphic than we necessarily have an $\mathbb{R}$-equivariant homeomorphism of their primitive ideal spaces that preserves the dimension of the target Hilbert spaces. To see this, let $\Psi : A \to B$ be an $\mathbb{R}$-equivariant isomorphism of two C^*-dynamical systems; if π is a representation of B then $\pi \circ \Psi$ is a representation of A; the inverse of this map on representations is obvious. Therefore it preserves the finite dimensional representations and consequently $\hat{\mathcal{P}}_S(K)$ and $\hat{\mathcal{P}}_S(L)$ are $\mathbb{R}$-equivariantly homeomorphic $\square$

4.2.1.1 An example: $K_{\rho,T-c}$

In the case when the S-unit group $\mathcal{O}_S^*(K)$ of K is finite, for instance when $K = K_{\rho,T-c}$ as in Section 3.1.2, we can weaken the $\mathbb{R}$-equivariant isomorphism condition slightly on the above result:

Corollary 4.2.7. *If K is such that $\mathcal{O}_S^*(K)$ is finite then the conclusion of Prop. 4.2.5 holds if we only assume an $\mathbb{R}$-equivariant homeomorphism $\Phi : \mathrm{Prim}(\mathcal{A}_{K,S}) \to \mathrm{Prim}(\mathcal{A}_{L,R})$.*

We are able to weaken this condition because of an number theoretic fact about the S-idele group:

Proposition 4.2.8. *If $\mathcal{O}_S^* \subset K$ is finite then K_S^* is closed in $\mathbb{A}_S^*$ and, consequently, the action of $\mathbb{R}$ on Prim$(\mathcal{A}_{K,S}) \setminus \hat{\mathcal{P}}_S$ is trivial.*

Proof. Chose a sequence $f_n \in K^*$ that converges to $(a_\mathfrak{p}) \in \mathbb{A}_S^*(K)$ in the idele group. We will first show that $(a_\mathfrak{p}) \in K_S^*$. We may assume that the sequence of divisors $\mathrm{div}_S(f_n) = ad(f_n)$ is constant, because the continuity of $ad : \mathbb{A}_S(K) \to \mathcal{D}_S(K)$ means that $ad(f_n) \to ad(a_\mathfrak{p})$, but since $\mathcal{D}_S(K)$ is discrete convergence means that the sequence eventually becomes constant. This implies that $ad(f_k) = ad(a_\mathfrak{p})$ for $k > N$ for some N. Now fix an $f = f_{k_0}$

for some $k_0 > N$. We also have that $f^{-1}f_n \to f^{-1}(a_\mathfrak{p})$ and therefore that $ad(f^{-1}f_n) \to ad(f^{-1}(a_\mathfrak{p}))$. But $ad(f^{-1}f_n) = 0$, which means that $f^{-1}f_n \in \mathcal{O}_S^*(K)$. By assumption this is finite, so its diagonal embedding in $\mathbb{A}_S^*(K)$ is finite and therefore closed (recall that $\mathbb{A}_S^*(K)$ is Hausdorff), so $f^{-1}(a_\mathfrak{p})$ must also be in the diagonal embedding $\mathcal{O}_S^*(K)$. In other words, $(a_\mathfrak{p}) \in K_S^*$.

Now let us reconsider the symbols

$$\Gamma_R := \{ad(a_\mathfrak{p}) : (a_\mathfrak{p}) \in \overline{K_S^*},\ a_\mathfrak{p} = 1 \text{ for } \mathfrak{p} \notin R\}.$$

Since $K_S^* \subset \mathbb{A}_S^*(K)$ is closed, Γ_R is a subset of the diagonal embedding of K^* into the S-ideles. If R is non-empty then at least one place has $a_\mathfrak{p} = 1$, and therefore they all do. So $\Gamma_R = \{1\}$.

Now let $[a, \chi] \in X_{K,S} \times \hat{\mathcal{D}}_S(K) /_\sim \cong \mathrm{Prim}(\mathcal{A}_{K,S})$ and recall that $\sigma_t([a,\chi]) = [a, N_S(\cdot)^{it}\chi]$. If $[a,\chi] \notin \hat{\mathcal{P}}_S(K)$ then we may take a to be represented by an element of the form $((a_\mathfrak{p}), 1)$ where $(a_\mathfrak{p})$ is non-zero. We have that $[a,\chi] \sim [a, N_S(\cdot)^{it}\chi]$ since $\mathcal{D}_{S,a} = \Gamma_{R_a}$ is trivial. □

Proof. Of Cor. 4.2.7

Let $\Phi : \mathrm{Prim}(\mathcal{A}_{K,S}) \to \mathrm{Prim}(\mathcal{A}_{L,R})$ be an $\mathbb{R}$-equivariant homeomorphism. $\mathbb{R}$-equivariance will preserve orbits, so that if $[a,\chi] \in \hat{\mathcal{P}}_S(K)$ then $\Phi([a,\chi])$ must have an infinite orbit in $\mathcal{A}_{L,R}$, and therefore we have that $\Phi([a,\chi])$ must be in $\hat{\mathcal{P}}_R(L)$. This yields that $\Phi(\hat{\mathcal{P}}_S(K)) \subset \hat{\mathcal{P}}_R(L)$. By symmetry we have that $\Phi(\hat{\mathcal{P}}_S(K)) = \hat{\mathcal{P}}_R(L)$. □

This result exactly mirrors the situation for number fields where the imaginary quadratic fields also have a trivial $\mathbb{R}$ action on the primitive ideal space of their

Bost-Connes systems.

4.2.1.2 Proof of Theorem 4.2

For the rest of the section we will use $b^{\mathbb{Z}}$ to mean the free abelian group with a single generator, namely, b. This is isomorphic to $\mathbb{Z}$, and its dual is group isomorphic to $\mathbb{T}$. Much of the work for the above Theorem 4.2 has already been done. Because of Prop. 4.2.5 we only need to show that: (1) an $\mathbb{R}$-equivariant isomorphism between the dual groups $\hat{\mathcal{P}}_S(K) \to \hat{\mathcal{P}}_R(L)$ implies an S-norm preserving map on the principal S-divisors; and (2) that an $\mathbb{R}$-equivariant homeomorphism implies such an isomorphism. We start with the former assertion:

Proposition 4.2.9. *If $\hat{\phi} : \hat{\mathcal{P}}_S(K) \to \hat{\mathcal{P}}_R(L)$ is an $\mathbb{R}$-equivariant group isomorphism then the induced isomorphism $\phi : \mathcal{P}_R(L) \to \mathcal{P}_S(K)$ preserves the norm.*

Proof. Let σ_t^K and σ_t^L be the $\mathbb{R}$-action on $\hat{\mathcal{P}}_S(K)$ and $\hat{\mathcal{P}}_S(L)$, respectively, and Let $b \in \mathcal{P}_R(L)$ be a generator and let $\mathrm{ev}_b : \hat{\mathcal{P}}_R \to \mathbb{T} = \hat{b^{\mathbb{Z}}}$ be the evaluation map. We have that $\mathrm{ev}_b(\sigma_t^L(\upsilon)) = N_R(b)^{it}\upsilon(b)$ for all $\upsilon \in \hat{\mathcal{P}}_R(L)$. $\mathbb{R}$-equivariance tells us that

$$\hat{\phi}(\sigma_t^K(\chi)) = \sigma_t^L(\hat{\phi}(\chi))$$

for all $\chi \in \hat{\mathcal{P}}_S(K)$. Then we have that

$$\mathrm{ev}_b(\hat{\phi}(\sigma_t^K(\chi))) = \hat{\phi} \circ \sigma_t^K(\chi)(b) = N_S(\phi(b))^{it}\chi \circ \phi(b)$$

On the other hand, we have that

$$\mathrm{ev}_b(\sigma_t^L(\hat{\phi}(\chi))) = N_R(b)^{it}\hat{\psi}(\chi)(b)$$

Which implies $N_R(b) = N_S(\phi(b))$ $\square$

We shall proceed by comparing the dual of the principal S-divisors to a product of circle groups $\mathbb{T}$. Let us be precise by what we mean by the dynamics on a product $\prod_j \mathbb{T}_J$: Choose an sequence $r_j \in \mathbb{R}_+$ and let $(\prod_j \mathbb{T}_j, \prod_j r_j)$ denote a dynamical system on $\prod_j \mathbb{T}_j$ by:

$$\sigma_t((x_j)) = (r_j^{it} x_j)$$

Lemma 4.2.10. *Let $a \in \mathcal{P}_S(K)$ be such that $N_S(a)$ is a generator for $N_S(\mathcal{P}_S(K))$. Then we have that $\hat{\mathcal{P}}_S(K)$ is $\mathbb{R}$-equivariantly isomorphic to*

$$(\mathbb{T}_a \times \mathbb{T}^\infty, N_{S_K}(a) \times 1)$$

Proof. The image of the S-norm of K on $\mathcal{P}_S(K)$ must be a non-trivial subgroup of $\mathbb{Z}$, and so isomorphic to $\mathbb{Z}$ itself. We will denote this image by $a^{\mathbb{Z}}$. This gives us a split exact sequence

$$0 \to \ker N_S \to \mathcal{P}_S(K) \to a^{\mathbb{Z}} \to 0$$

So we have that $\mathcal{P}_S(K) \cong a^{\mathbb{Z}} \oplus \ker N_S$. Taking the dual gives us that $\hat{\mathcal{P}}_S(K) \cong \mathbb{T}_a \times \mathbb{T}^\infty$. To be explicit, let $\chi \in \hat{\mathcal{P}}_S(K)$. Then

$$\chi \mapsto \chi(a) \times \Big(\prod_{\tilde{D}} \chi(\tilde{D})\Big) \in \mathbb{T}_a \times \mathbb{T}^\infty,$$

where $\tilde{D}$ runs through a set of all generators of $\ker N_S$, is a bijective group homomorphism. Its inverse is obvious. To see that this isomorphism is R-equivariant, let $D \in \mathcal{P}_S(K)$. Write may write $D = (D_a \cdot a) \oplus D_0$ for $D_0 \in \ker N_S$, so that $N_S(D) = N_S(a)^{D_a}$. This yields

$$\sigma_t(\chi)(D) = \left(N_S(a)^{it}\chi(a)\right)^{D_a} \cdot \chi(D_0)$$

Hence $(\hat{\mathcal{P}}_S(K), \sigma_t)$ is R-equivariantly isomorphic to $(\mathbb{T}_a \times \mathbb{T}^\infty, N_S(a) \times \prod_{j=1}^\infty 1)$.

$\square$

Proof. of Theorem 4.2

Recall that we need to show that:

(1) an $\mathbb{R}$-equivariant isomorphism between the dual groups $\hat{\mathcal{P}}_S(K) \to \hat{\mathcal{P}}_R(L)$ implies an S-norm preserving map on the principal S-divisors; and

(2) that an $\mathbb{R}$-equivariant homeomorphism implies such an isomorphism.

We have shown (1) in Prop. 4.2.9, so we prove (2) here. Let $\varphi : \hat{\mathcal{P}}_S(K) \to \hat{\mathcal{P}}_R(L)$ be an $\mathbb{R}$-equivariant homeomorphism from, say, Prop. 4.2.5. By the previous lemma we have that

$$\hat{\mathcal{P}}_S(K) \cong \left(\mathbb{T}_a \times \mathbb{T}^\infty, N_S(a) \times 1\right)$$

$$\hat{\mathcal{P}}_R(L) \cong \left(\mathbb{T}_b \times \mathbb{T}^\infty, N_R(b) \times 1\right)$$

The non-trivial part of the $\mathbb{R}$-action corresponds to a map $x \mapsto N_S(a)^{it}x$ for $x = e^{i\theta} \in \mathbb{T}_a$. This corresponds to a rotation of the circle group $\mathbb{T}_a$ by the irrational number $\log N_S(a)$. It is well-known (for instance, in Theorem 0.14

of [35]) that the only closed subgroups of $\mathbb{T}$ are either finite or all of $\mathbb{T}$. Since $\log N_S(a)$ is irrational, the orbits of this action are dense subgroups, and each orbit can be identified with $\mathbb{T}_a \times \{x\}$ for some $x \in \mathbb{T}^\infty$. This gives us the orbit decomposition

$$\hat{\mathcal{P}}_S(K) \cong \bigsqcup_{x \in \mathbb{T}^\infty} \mathbb{T}_a \times \{x\}$$

$$\hat{\mathcal{P}}_R(L) \cong \bigsqcup_{y \in \mathbb{T}^\infty} \mathbb{T}_b \times \{y\}$$

By $\mathbb{R}$-equivariance we have that $\varphi(\mathbb{T}_a \times \{1\}) = \mathbb{T}_b \times \{y\}$ for some $y \in \mathbb{T}^\infty$ and therefore an $\mathbb{R}$-equivariant homeomorphism

$$\bar{\varphi} : \mathbb{T}_a \to \mathbb{T}_b$$

Note that $\mathbb{R}$-equivariance implies that

$$\bar{\varphi}(1) = \frac{\overline{\varphi}(N_{S_K}(a)^{it})}{N_{S_L}(b)^{it}}$$

For all $t \in \mathbb{R}$, in particular for $t = 2\pi$. Let $x = N_{S_K}(a)^{2\pi i}$ and $y = N_{S_L}(b)^{2\pi i}$. Then $\bar{\varphi}(1)^{-1}\bar{\varphi}(x^n) = y^n$ for all $n \in \mathbb{Z}$. This action by $\mathbb{Z}$ on the x again has dense orbits. Hence we have an $\mathbb{R}$-equivariant group isomorphism. If $\tau : \mathbb{T}^\infty \to \mathbb{T}^\infty$ is any group isomorphism and $\phi(x) = \bar{\varphi}(1)^{-1}\bar{\varphi}(x)$, then $\phi \times \tau : \hat{\mathcal{P}}_S(K) \to \hat{\mathcal{P}}_R(L)$ is an $\mathbb{R}$-equivariant group isomorphism and by Prop. 4.2.9 we have norm-preserving isomorphism between the respective principal S-divisor groups. $\square$

Chapter 5

Arithmetic equivalences interpreted as equivalences of dynamical systems.

Let K and L be global function fields with non-empty finite subsets of primes S and R, respectively. We turn our attention away from C^* algebras and their invariants and towards the study of underlying monoid of the Bost-Connes type systems:

$$Y_{K,S} := \hat{\mathcal{O}}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \hat{\mathcal{O}}_S^*(K).$$

As described in Chapter 3, this is a topological monoid on which acts the monoid of effective S-divisors $\mathcal{D}_S^+(K)$. We will study this object as an abstract topological dynamical system.

Definition 5.1. *By a* ***topological dynamical system*** $G \curvearrowright X$ *we mean a group (or semigroup or monoid) G acting continuously on a topological space*

X.

There a number of natural equivalences of such systems, namely:

Definition 5.2. *Two topological dynamical systems $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ are **orbit equivalent** if there exists a homeomorphism $\Phi : X_1 \to X_2$ such that $\Phi(G_1 \cdot x) = G_2 \cdot \Phi(x)$ for all $x \in X_1$.*

Definition 5.3. *Two topological dynamical systems $G_1 \curvearrowright X_1$ and $G_2 \curvearrowright X_2$ are **conjugate** if there exists a homeomorphism $\Phi : X_1 \to X_2$ together with a group/semigroup/monoid isomorphism $\phi : G_1 \to G_2$ such that $\Phi(g \cdot x) = \phi(g) \cdot \Phi(x)$ for all $g \in G_1$ and $x \in X_1$.*

It is clear that two systems are orbit equivalence if they are conjugate.

In our particular case we note that the effective S-divisors may be embedded within $Y_{K,S}$ by considering their action on the unit $[1, 1] \in Y_{K,S}$. It is reasonable to ask for a homeomorphism that respects the algebraic structure:

Definition 5.4. *$\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ and $\mathcal{D}_R^+(L) \curvearrowright Y_{L,R}$ are **algebraically equivalent** if there exists a topological isomorphism $\Phi : Y_{K,S} \to Y_{L,R}$ that restricts to a monoid isomorphism on $\mathcal{D}_S^+(K) \to \mathcal{D}_R^+(L)$.*

It is the aim of this chapter is to give arithmetic interpretations to these equivalences of topological dynamical systems. In what follows we will describe three "bespoke" arithmetic equivalences: an S-R Reciprocity isomorphism, a Finite Reciprocty isomorphism, and an L-function Isomorphism. Each may fall short of an isomorphism of function fields $K \cong L$, or even commutative square where the horizontal arrows are isomorphisms:

$$\begin{array}{ccc} \mathbb{A}_S^*(K) & \xrightarrow{\quad\tilde{\phi}\quad} & \mathbb{A}_R^*(L) \\ {\scriptstyle \vartheta_{K^{\mathrm{ab}}(S)}}\downarrow & & \downarrow{\scriptstyle \vartheta_{L^{\mathrm{ab}}(R)}} \\ \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) & \xrightarrow{\psi} & \mathrm{Gal}(L^{\mathrm{ab}}(R)/L) \end{array}$$

The reasons for this failure will be discussed in Chapter 6. Nevertheless, each equivalence recovers a portion of the class field theoretic information contained therein in a manner that we will now explain. We start with the S-R Reciprocity isomorphism.

Definition 5.5. *Let K and L be global function fields over $\mathbb{F}_q$ together with respective finite subsets of primes S and R. By an S-R* ***Reciprocity Isomorphism*** *between K and L we mean the following data:*

1. *A monoid isomorphism $\phi : \mathcal{D}_S^+(K) \to \mathcal{D}_R^+(L)$; and*
2. *An isomorphism of topological groups ψ : Gal$(K^{\mathrm{ab}}(S)/K) \to$ Gal$(L^{\mathrm{ab}}(R)/L)$; and*
3. *Splits $s_K : \mathcal{D}_S^+(K) \to \mathbb{A}_S^*(K)$ and $s_L : \mathcal{D}_R^+(L) \to A_R^*(L)$*

such that

(a) For all $D \in \mathcal{D}_S^+(K)$ we have that

$$\psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D) = \vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(D).$$

(b) For all $\mathfrak{p} \in \mathcal{S}_K \setminus S$, we have that

$$\psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ i(\mathcal{O}_{\mathfrak{p}}^*) = \vartheta_{L^{\mathrm{ab}}(R)} \circ i(\mathcal{O}_{\phi(\mathfrak{p})}^*).$$

Condition (a) is akin to asking that the *idealic* artin symbol commutes with ψ. While Condition (b) might appear esoteric, when combined with Lemma 2.2.17 we see that it can be interpreted as requiring that the Galois group of the maximal extension of K in $K^{\mathrm{ab}}(S)$ that is unramified at $\mathfrak{p}$ is mapped by ψ to the Galois group of the maximal extension of L in $L^{\mathrm{ab}}(R)$ unramified at $\phi(\mathfrak{p})$.

Taken alone, Condition (a) is a strong restriction on the splits s_K and s_L. Since the choice of a split s_K (respectively, s_L) essentially mean a choice of $(r_{\mathfrak{p}}) \in \hat{\mathcal{O}}_S^*(K)$ (respectively, $(t_{\mathfrak{q}}) \in \hat{\mathcal{O}}_R^*(L)$), we have that $s_K(0) = (r_{\mathfrak{p}})$ (respectively, $s_L \circ \phi(0) = s_L(0) = (t_{\mathfrak{q}})$, noting that a monoid homomorphism preserves the identity) and this means that condition (a) requires that

$$\vartheta_{L^{\mathrm{ab}}(R)}(t_{\mathfrak{q}}) = \psi \circ \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}).$$

Yet a consequence of Condition (b) is that

$$\psi \circ \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}) \in \vartheta_{L^{\mathrm{ab}}(R)}(\hat{\mathcal{O}}_R^*(L)).$$

So Conditions (a) and (b) are independent, but where Condition (b) is met the necessary requirements for Condition (a) are relaxed.

The meaning of the next arithmetic equivalence, the Finite Reciprocity Isomorphism, is perhaps more obvious. If $N \subset \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ is a subgroup then K^N will denote the field fixed by N.

Definition 5.6. *By a **Finite Reciprocity Isomorphism** between K and L we mean the following data:*

1. *A monoid isomorphism $\phi : \mathcal{D}_S^+(K) \to \mathcal{D}_R^+(L)$*

2. An isomorphism of topological groups $\psi : Gal(K^{\text{ab}}(S)/K) \to Gal(L^{\text{ab}}(R)/L)$

such that for every subgroup $N \subset Gal(K^{\text{ab}}(S)/K)$ we have that ϕ is a bijection between the unramified primes of K^N/K and $L^{\psi(N)}/L$ such that $\psi(Frob_{K^N/K}(\mathfrak{p})) = Frob_{L^{\psi(N)}/L}(\phi(\mathfrak{p}))$.

And finally we have the L-function Isomorphism:

Definition 5.7. *By a L-**function Isomorphism** between K and L we mean an isomorphism of topological groups $\psi : Gal(K^{\text{ab}}(S)/K) \to Gal(L^{\text{ab}}(R)/L)$ such that*

$$L_{K,\chi}(z) = L_{L,\hat{\psi}(\chi)}(z) \text{ for all } z \in \mathbb{C}$$

For all characters $\chi : Gal(K^{\text{ab}}(S)/K) \to \mathbb{T}$, where $\hat{\psi}(\chi) = \chi \circ \psi^{-1}$.

The main result of this chapter is

Theorem 5.1. *The three equivalences of topological dynamical systems and the three arithmetic equivalences are all equivalent to each other.*

This result is a generalization of the results and methods of [11] and [13] to global function fields excluding a finite set of primes, which we have shown in Chapter 4 to be a Bost-Connes type systems associated to function fields that has stronger analogies with the number field case.. In Section 5.1 we will show that each of the dynamical system equivalences are equivalent with each. In Section 5.2 will show that an S-R Reciprocity Isomorphism is equivalent to our three dynamical system equivalences. Finally, in Section 5.3 we will show that the notions of an S-R Reciprocity Isomorphism, a Finite Reciprocity Isomorphism, and an L-function Isomorphism are all equivalent as well.

5.1 Topological Dynamics of $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$

Let K (respectively L) be a global function field with S (respectively R) a non-empty finite set of primes. If we have an algebraic equivalence of the dynamical systems

$$\Phi : \mathcal{D}_S^+(K) \curvearrowright Y_{K,S} \to \mathcal{D}_S^+(L) \curvearrowright Y_{L,R}$$

and we denote by ϕ its restriction to $\mathcal{D}_S^+(K)$ then it is clear that this implies orbit equivalence. We will show in Prop. 5.1.2 that an orbit equivalence between $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ and $\mathcal{D}_S^+(L) \curvearrowright Y_{L,R}$ implies that the two systems are conjugate, and in Prop. 5.1.3 that conjugacy implies an algebraic isomorphism. We start with an algebraic consequence of orbit equivalence:

Lemma 5.1.1. *Let K and L be global function fields and let S and R be finite subsets of primes of K and L, respectively. If $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ is orbit equivalent with $\mathcal{D}_R^+(L) \curvearrowright Y_{L,R}$ then the unit of $Y_{K,S}$ is invertible in $Y_{L,R}$.*

Proof. Let $\Phi : Y_{K,S} \to Y_{L,R}$ be the orbit equivalence and let $[1,1] \in Y_{K,S}$ be the unit. By Lemma 3.1.13 we know that $\mathcal{D}_S^+(K) \cdot [1,1]$ is dense in $Y_{K,S}$. Since Φ is a homeomorphism we also have that $\Phi(\mathcal{D}_S^+(K) \cdot [1,1]) = \mathcal{D}_R^+(L) \cdot \Phi([1,1])$ is dense in $Y_{L,R}$. Say that $\Phi([1,1]) = [(a_\mathfrak{p}), \alpha]$ is not invertible. This means that there exists a prime $\mathfrak{q} \in \mathcal{S}_L \setminus R$ such that $a_\mathfrak{q} \in \mathcal{O}_\mathfrak{q} \setminus \mathcal{O}_\mathfrak{q}^*$, id est, $v_\mathfrak{q}(a_\mathfrak{q}) > 0$. This property is persistent through the action $D \cdot [(a_\mathfrak{p}), \alpha] = [(d_\mathfrak{p} a_\mathfrak{p}), \vartheta_{L^{\mathrm{ab}}(R)}(d_\mathfrak{p})^{-1}\alpha]$ as $(d_\mathfrak{p}) = s(D)$ has $v_\mathfrak{p}(d_p) \geq 0$ for an effective divisor. The property persists even after taking the closure, as the adelic component of any converging sequence in $\mathcal{D}_R^+(L) \cdot \Phi([1,1])$ must converge to an element with positive $\mathfrak{q}$-valuation. This contradicts that $\mathcal{D}_R^+(L) \cdot \Phi([1,1])$ is dense. $\square$

As it is clear that if two systems are conjugate they are also orbit equivalent, the next proposition shows that, for our arithmetic systems, the two concepts are equivalent.

Proposition 5.1.2. *If $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ is orbit equivalent with $\mathcal{D}_R^+(L) \curvearrowright Y_{L,R}$ then they are conjugate.*

Proof. Since the two systems are orbit equivalent, for every $D \in \mathcal{D}_S^+(K)$ there exists a $F \in \mathcal{D}_R^+(L)$ such that $\Phi(D \cdot [1,1]) = F \cdot \Phi([1,1])$, since $\Phi([1,1])$ is invertible this $F = \phi(D)$ is unique. By considering Φ^{-1} we can find another such map $\phi^{-1} : \mathcal{D}_R^+(L) \to \mathcal{D}_S^+(K)$ and these maps are mutually inverse. By Lemma 3.1.13 we know that the effective divisors are dense, so it will suffice to show that the map ϕ respects divisor addition and carries the zero-divisor of $\mathcal{O}_S(K)$ to that of $\mathcal{O}_R(L)$. Let $\mathfrak{p} \in \mathcal{S}_K \setminus S$. We will first show that there is a prime $\mathfrak{q} \in \mathcal{S}_L \setminus R$ such that for any effective S-divisor $D \in \mathcal{D}_S^+(K)$ we have that $\phi(\mathfrak{p} + D) = \mathfrak{q}_D + \phi(D)$ for $\mathfrak{q}_D \in \mathcal{S}_L \setminus R$. Note that

$$\phi(\mathfrak{p}+D)\cdot\Phi([1,1]) = \Phi((\mathfrak{p}+D)\cdot[1,1]) \in \mathcal{D}_R^+(L)\cdot\Phi(D\cdot[1,1]) = \mathcal{D}_R^+(L)\cdot\phi(D)\cdot\Phi([1,1])$$

since $\Phi([1,1])$ is invertible this implies that $\phi(\mathfrak{p} + D) \in \mathcal{D}_R^+(L) \cdot \phi(D)$ and therefore there is an $F_D \in \mathcal{D}_R^+(L)$ such that $\phi(\mathfrak{p} + D) = F_D + \phi(D)$. We shall see that $F_D \in \mathcal{S}_L \setminus R$; if it is not, then we can let G and H be non-zero effective

R-divisors of L such that $F = G + H$. Then we must have

$$
\begin{aligned}
(\mathfrak{p} + D) \cdot [1,1] &= \Phi^{-1}(\phi(\mathfrak{p} + D) \cdot \Phi([1,1])) \\
&= \Phi^{-1}(F_D \cdot \phi(D) \cdot \Phi([1,1])) \\
&= \Phi^{-1}((G + H) \cdot \phi(D) \cdot \Phi([1,1]) \\
&= (\tilde{G} + \tilde{H}) \cdot \Phi^{-1}(\phi(D) \cdot \Phi([1,1])) \\
&= (\tilde{G} + \tilde{H} + D) \cdot [1,1] \\
\implies \mathfrak{p} = \tilde{G} + \tilde{H}
\end{aligned}
$$

for some (potentially zero) $\tilde{G}, \tilde{H} \in \mathcal{D}_S^+(K)$. Let $x = \Phi^{-1}((H + \phi(D)) \cdot \Phi([1,1]))$ and $y = \Phi(x)$. Moreover, say that y is represented by a $[(b_\mathfrak{q}), \beta] \in Y_{L,R}$ with $(b_\mathfrak{q}) \neq 0$. Clearly $s_L(G) \cdot (b_\mathfrak{q}) \neq (b_\mathfrak{q})$ and $G \cdot y \neq y$, and so $x \neq \tilde{G} \cdot x$, so we have that $\tilde{G}$ is non-zero. We may repeat this argument for $\tilde{H}$. But then we have that $\mathfrak{p}$ is the sum of non-zero S-divisors, which is a contradiction. But if F_D cannot be written as the sum of non-zero divisors then $F_D \in \mathcal{S}_L \setminus R$.

Using this, we may show that for $D = \sum_{\mathfrak{p} \notin S} D_\mathfrak{p} \mathfrak{p}$ we have that $\phi(D) = \sum_{\mathfrak{p} \notin S} D_\mathfrak{p} \phi(\mathfrak{p})$ for some $\mathfrak{q}_i \in \mathcal{S}_L \setminus R$. We will first show that $\phi(n \cdot \mathfrak{p}) = n \cdot \phi(\mathfrak{p})$. At the least we have already shown that $\phi(n \cdot \mathfrak{p}) = \sum_{j=1}^n \mathfrak{q}_j$. This means that

$$
\left(\mathcal{D}_R^+(L) + \phi(n \cdot \mathfrak{p})\right) \cdot \Phi([1,1]) \subset \left(\mathcal{D}_R^+(L) + \mathfrak{q}_j\right) \cdot \Phi([1,1])
$$

for every $j = 1 \ldots n$. But applying Φ^{-1} yields that

$$
\mathcal{D}_S^+(K) + (n \cdot \mathfrak{p}) \subset \mathcal{D}_S^+(K) + \phi^{-1}(\mathfrak{q}_j)
$$

For effective divisors the only choice for $\phi^{-1}(\mathfrak{q}_j)$ is $\mathfrak{p}$, and so we have that

$\phi(n \cdot \mathfrak{p}) = n \cdot \phi(\mathfrak{p})$.

At this point, we have that $\phi(\mathfrak{p} + D) = \mathfrak{q}_D + \phi(D)$ and $\phi(n \cdot \mathfrak{p}) = n \cdot \phi(\mathfrak{p})$, with $F \in \mathcal{S}_L \setminus R$. We will now show that $\mathfrak{q}_D = \phi(\mathfrak{p})$. Returning to the case of $D = \sum_{\mathfrak{p} \notin S} D_{\mathfrak{p}} \mathfrak{p}$ we have that

$$\phi(D) = D' := \sum_{\phi(\mathfrak{p}) \notin R} D'_{\phi(\mathfrak{p})} \phi(\mathfrak{p}),$$

note that $D'_{\phi(\mathfrak{p})}$ is the unique natural number such that

1. $\mathcal{D}^+_R(L) + \phi(D) \subset \mathcal{D}^+_R(L) + (D'_{\phi(\mathfrak{p})} \cdot \phi(\mathfrak{p}))$; and
2. $\mathcal{D}^+_R(L) + \phi(D) \not\subset \mathcal{D}^+_R(L) + ((D'_{\phi(\mathfrak{p})} + 1) \cdot \phi(\mathfrak{p}))$.

But multiplying by $\Phi([1,1])$ and applying Φ^{-1} shows that

$$\mathcal{D}^+_S(K) + D \subseteq \mathcal{D}^+_S(K) + (D'_{\phi(\mathfrak{p})} \cdot \mathfrak{p}) \text{ and } \mathcal{D}^+_S(K) + D \not\subset \mathcal{D}^+_S(K) + ((D'_{\phi(\mathfrak{p})} + 1) \cdot \mathfrak{p})$$

Which forces us to conclude that $D'_{\phi(\mathfrak{p})} = D_{\mathfrak{p}}$. In particular we have that $\phi(\mathfrak{p} + D) = \phi(\mathfrak{p}) + \phi(D)$ as desired. Altogether we have shown for that for $D = \sum_{\mathfrak{p} \notin S} D_{\mathfrak{p}} \mathfrak{p}$ we have that

$$\phi(D) = \sum_{\phi(\mathfrak{p}) \notin R} D_{\mathfrak{p}} \phi(\mathfrak{p}),$$

in other words ϕ is a monoid isomorphism. $\square$

If in addition to orbit equivalence we have that for every effective divisor $D \in \mathcal{D}^+_S(K)$ and $x \in Y_{K,S}$ we can find an effective divisor $F \in \mathcal{D}^+_R(L)$ such that $N_R(F) = N_S(D)$ and $\Phi(D \cdot x) = F \cdot \Phi(x)$ then our monoid isomorphism in the above proof also preserves the norm. If we start from the assumption that two such systems are conjugate it is a slightly easier task to return to an algebraic equivalence.

Proposition 5.1.3. *If $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ and $\mathcal{D}_R^+(L) \curvearrowright Y_{L,R}$ are conjugate then we have an algebraic equivalence between them.*

Proof. Let $\Phi : Y_{K,S} \to Y_{L,R}$ be the homeomorphism with $\phi : \mathcal{D}_S^+(K) \to \mathcal{D}_R^+(L)$ its related monoid isomorphism. By Lemma 5.1.1 we know that $\Phi([1,1])$ is invertible. Call it's inverse u and define a new homeomorphism by $\Phi'(x) = \Phi(x)u$. We show that this map is a monoid homomorphism when restricted to the effective divisors: Let $D, F \in \mathcal{D}_S^+(K)$ and then

$$\begin{aligned}
\Phi'((D+F)\cdot[1,1]) &= \Phi((D+F)\cdot[1,1])u \\
&= (\phi(D)+\phi(F))\cdot(\Phi([1,1])u) \\
&= \big(\phi(D)\cdot\Phi([1,1])u\big)\cdot\big(\phi(F)\cdot\Phi([1,1])u\big) \\
&= \Phi'(D)\Phi'(F)
\end{aligned}$$

Recall that $\iota(D) = D\cdot[1,1]$. By Lemma 3.1.13 we know that effective divisors are dense, indeed the description of the action of the divisors in Chapter 3 tell us that $(D+F)\cdot[1,1] = \iota(D)\cdot\iota(F)$. This is sufficient information for us to conclude that Φ' is an isomorphism between topological monoids. From the above it is also plain that $\Phi'(D\cdot[1,1]) = \phi(D)\cdot[1,1] \in Y_{L,R}$. □

5.2 S-R Reciprocity Isomorphism

We shall start recovering arithmetic information from the equivalence of topological dynamical systems described in the last section. In particular we shall prove in this section:

Theorem 5.2.1. *The following are equivalent:*

- *There is an algebraic equivalence between* $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ *and* $\mathcal{D}_R^+(L) \curvearrowright Y_{L,R}$.
- *There is an* S-R *Reciprocity isomorphism between* K *and* L.

5.2.1 Algebraic equivalence to S-R Reciprocity Isomorphism

We shall start by assuming algebraic equivalence, id est, that we have a topological monoid isomorphism $\Phi : Y_{K,S} \to Y_{L,R}$ (call its restriction to effective divisors $\phi = \Phi|_{\mathcal{D}_S^+(K)}$). We denote the isomorphism of Galois groups with the following:

Proposition 5.2.2. *Let* $\psi = \Phi|_{Y^*_{K,S}}$. *Then* ψ *is an isomorphism of topological groups* $Gal(K^{\text{ab}}(S)/K) \to Gal(L^{\text{ab}}(R)/L)$.

Proof. If $Y^*_{K,S}$ denotes the unit group of $Y_{K,S}$ then we have that $Y^*_{K,S} = \left(\hat{\mathcal{O}}^*_S \times \mathrm{Gal}(K^{\text{ab}}(S)/K)\right) \big/ \hat{\mathcal{O}}^*_S$. We have a map $\mathrm{Gal}(K^{\text{ab}}(S)/K) \cong Y^*_{K,S}$ by $\alpha \to [1, \alpha]$. □

Having assembled the monoid isomorphism $\phi : \mathcal{D}_S^+(K) \to \mathcal{D}_R^+(L)$ and the ismorphism of topological groups $\psi : \mathrm{Gal}(K^{\text{ab}}(S)/K) \to \mathrm{Gal}(L^{\text{ab}}(R)/L)$, we proceed to show that they meet Condition (b). As described in the introduction this will make it easier for us to show that we also have Condition (a).

We shall need some additional notation: For a subset $P \subset \mathcal{S}_K \setminus S$ let P^c denote the set $\mathcal{S}_K \setminus (S \cup P)$. Since ϕ is a monoid isomorphism on effective S-divisors it is a bijection between $\mathcal{S}_K \setminus S$ and $\mathcal{S}_L \setminus R$, so we may consider $\phi(P)$ a subset of $\mathcal{S}_L \setminus R$. Let $N_{\vartheta^{\mathrm{ab}}_K(S)}(P)$ denote the subgroup:

$$N_{\vartheta^{\mathrm{ab}}_K(S)}(P) := \vartheta_{K^{\mathrm{ab}}(S)} \circ i(\prod_{\mathfrak{p} \in P} \mathcal{O}^*_{\mathfrak{p}})$$

Additionally, let $(\delta^P_{\mathfrak{p}})$ denote the element in $\hat{\mathcal{O}}_S$ indexed with:

$$\delta^P_{\mathfrak{p}} = \begin{cases} 1 & \mathfrak{p} \in P \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5.2.3. *We have that $\psi(N_{\vartheta_{K^{\mathrm{ab}}(S)}}(P)) = N_{\vartheta_{L^{\mathrm{ab}}(R)}}(\phi(P))$, in particular, $\psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ i(\mathcal{O}^*_{\mathfrak{p}}) = \vartheta_{L^{\mathrm{ab}}(R)} \circ i(\mathcal{O}^*_{\phi(\mathfrak{p})})$ for all $\mathfrak{p} \in \mathcal{S}_K \setminus S$.*

Proof. For any set $P \subset \mathcal{S}_K \setminus S$ define the function

$$\mu_P : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to Y_{K,S} \cdot [(\delta^P_{\mathfrak{p}}), 1] \text{ by } \alpha \mapsto \alpha \cdot [(\delta^P_{\mathfrak{p}}), 1]$$

where the multiplication occurs as elements in the monoid $Y_{K,S}$. Also for a $Q \subset \mathcal{S}_L \setminus R$ we define

$$\mu_Q : \mathrm{Gal}(L^{\mathrm{ab}}(R)/L) \to Y_{L,R} \cdot [(\delta^Q_{\mathfrak{q}}), 1]$$

in the same way. The next lemma allows us to state that $\Phi([(\delta^{P^c}_{\mathfrak{p}}), 1]) = [(\delta^{\phi(P)^c}_{\mathfrak{q}}), 1]$, this yields that Φ restricts to an isomorphism

$$Y_{K,S} \cdot [(\delta P^c{}_{\mathfrak{p}}), 1] \to Y_{L,R} \cdot [(\delta^{\phi(P)^c}_{\mathfrak{q}}), 1].$$

Using the aforementioned lemma we can conclude that $\Phi \circ \mu_{P^c} = \mu_{\phi(P)^c} \circ \psi$. In particular we have that

$$\psi\big(\mu_{P^c}^{-1}([(\delta_{\mathfrak{p}}^{P^c}), 1])\big) = \mu_{\phi(P^c)}^{-1}\big([(\delta_{\mathfrak{q}}^{\phi(P)^c}), 1]\big).$$

An element $\alpha \in \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ lies in $\mu_{P^c}^{-1}([(\delta_{\mathfrak{p}}^{P^c}), 1])$ if and only if there exits a $(r_{\mathfrak{p}}) \in \hat{\mathcal{O}}_S^*$ with $\alpha = \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})^{-1}$ and $(r_{\mathfrak{p}}) \cdot (\delta_{\mathfrak{p}}^{P^c}) = (\delta_{\mathfrak{p}}^{P^c})$, id est if $r_{\mathfrak{p}} = 1$ for $\mathfrak{p} \notin P$, id est $(r_{\mathfrak{p}}) \in i(\prod_{\mathfrak{p} \in P} \mathcal{O}_{\mathfrak{p}}^*)$. So we have that $\mu_{P^c}^{-1}([(\delta_{\mathfrak{p}}^{P^c}), 1]) = N_{\vartheta_K^{\mathrm{ab}}(S)}(P)$. By symmetry of our morphisms we also have that $\mu_{\phi(P)^c}^{-1}\big([(\delta_{\mathfrak{q}}^{\phi(P)^c}), 1]\big) = N_{\vartheta_L^{\mathrm{ab}}(R)}(\phi(P))$. Therefore $\psi(N_{\vartheta_K^{\mathrm{ab}}(S)}(P)) = N_{\vartheta_L^{\mathrm{ab}}(R)}(\phi(P))$ as desired. $\square$

It remains to be shown that Φ restricts to an isomorphism $Y_{K,S} \cdot [(\delta_{\mathfrak{p}}^{P}), 1] \to Y_{L,R} \cdot [(\delta_{\mathfrak{q}}^{\phi(P)^c}), 1]$. We actually prove a stronger statement:

Lemma 5.2.4. *For every $P \subset \mathcal{S}_K \setminus S$ we have that $\Phi([(\delta_{\mathfrak{p}}^{P}), 1])] = [(\delta_{\mathfrak{q}}^{\phi(P)}), 1]$*

Proof. Since $D \cdot [(a_{\mathfrak{p}}), \gamma] = [(\pi_{\mathfrak{p}}^{D_{\mathfrak{p}}} a_{\mathfrak{p}}), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)^{-1}\gamma]$ we have that

$$D \cdot Y_{K,S} = \{[(a_{\mathfrak{p}}), \gamma] \in Y_{K,S} : v_{\mathfrak{p}}(a_{\mathfrak{p}}) \geq D_{\mathfrak{p}}\}.$$

From this we can conclude that

$$[(\delta_{\mathfrak{p}}^{P}), 1] \cdot Y_{K,S} = \bigcap_{D \in \mathcal{D}_S^+(K), \mathrm{supp}(D) \cap P = \emptyset} D \cdot Y_{K,S}.$$

Applying Φ to both sides yields

$$\Phi([(\delta_{\mathfrak{p}}^{P}), 1]) \cdot Y_{L,R} = \bigcap_{D \in \mathcal{D}_S^+(K), \mathrm{supp}(D) \cap P = \emptyset} \phi(D) \cdot Y_{L,R} = [(\delta_{\mathfrak{q}}^{\phi(P)}), 1] \cdot Y_{L,R}.$$

Let $\Phi([(\delta_{\mathfrak{p}}^{P}), 1])$ be represented by $[(a_{\mathfrak{q}}), \alpha]$. The above equation implies that

$a_{\mathfrak{q}} = 0$ for $\mathfrak{q} \notin \phi(P)$ and $a_{\mathfrak{q}} \in \mathcal{O}^*_{\mathfrak{q}}$ otherwise. Since $[(\delta^P_{\mathfrak{p}}), 1]$ is idempotent $[(a_{\mathfrak{q}}), \alpha]$ must be as well. This implies that there exits a $(r_{\mathfrak{q}}) \in \hat{\mathcal{O}}^*_R$ with $\alpha^2 = \alpha \vartheta_{L^{\mathrm{ab}}(R)}(r_{\mathfrak{q}})^{-1}$ and $(a_{\mathfrak{q}})^2 = (a_{\mathfrak{q}} r_{\mathfrak{q}})$. This means that $\alpha = \vartheta_{L^{\mathrm{ab}}(R)}(r_{\mathfrak{q}})^{-1}$ and $(a_{\mathfrak{q}}) = (r_{\mathfrak{q}}) \cdot (\delta^{\phi(P)}_{\mathfrak{q}})$, in other words $[(a_{\mathfrak{q}}), \alpha] = [(\delta^{\phi(P)}_{\mathfrak{q}}), 1]$. $\square$

Now that we have shown that an algebraic equivalence yields monoid isomorphism that satisfy Condition (b) we move on to showing that we can meet Condition (a) as well. Recall that our definition of $Y_{K,S}$ was independent of choice of split s_K. First we will demonstrate that for a split s_K we can find a choice of split s_L that is compatible with Φ.

Proposition 5.2.5. *For a given split $s_K : \mathcal{D}^+_S(K) \to \mathbb{A}^*_S(K) \cap \hat{\mathcal{O}}_S(K)$ there exists a split $s_L : \mathcal{D}^+_R(L) \to \mathbb{A}^*_R(L) \cap \hat{\mathcal{O}}_R(L)$ such that*

$$\Phi([s_K(D), 1]) = [s_L(\phi(D)), 1].$$

Proof. We will define an s_L for a prime $\mathfrak{r} \in \mathcal{D}^+_S(K)$. This will suffice as it will extend linearly to all effective divisors. Let us begin by noting that

$$[s_K(\mathfrak{r}), 1] \cdot Y_{K,S} = \{[(a_{\mathfrak{p}}), \alpha] : v_{\mathfrak{r}}(a_{\mathfrak{r}}) > 0\}$$

Now $v_{\mathfrak{r}}(a_{\mathfrak{r}}) = 0$ is equivalent to $[(a_{\mathfrak{p}}), \alpha] \cdot [(\delta^{\{\mathfrak{r}\}}_{\mathfrak{p}}), 1] \cdot Y_{K,S} = [(\delta^{\{\mathfrak{r}\}}_{\mathfrak{p}}), 1] \cdot Y_{K,S}$. Use $[(b_{\mathfrak{q}}), \beta]$ to denote $\Phi([(a_{\mathfrak{p}}), \alpha])$ and apply Φ: then $v_{\mathfrak{r}}(a_{\mathfrak{r}}) = 0$ implies that $[(b_{\mathfrak{q}}), \beta] \cdot [(\delta^{\{\phi(\mathfrak{r})\}}_{\mathfrak{q}}), 1] \cdot Y_{L,R} = [(\delta^{\{\phi(\mathfrak{r})\}}_{\mathfrak{q}}), 1] \cdot Y_{L,R}$, which finally yields that:

$$\Phi([s_K(\mathfrak{r}), 1]) \cdot Y_{L,R} = \{[(b_{\mathfrak{q}}), \beta] : v_{\phi(\mathfrak{r})}(b_{\phi(\mathfrak{r})}) > 0\}.$$

Therefore we can represent $\Phi([s_K(\mathfrak{r}), 1])$ by a $[(b_{\mathfrak{q}}), \beta]$ with $v_{\phi(\mathfrak{r})}(b_{\phi(\mathfrak{r})}) = 1$.

Additionally, applying Φ to and again invoking Lemma 5.2.4 on the following equation

$$[s_K(\mathfrak{r}), 1] \cdot [(\delta_{\mathfrak{p}}^{\{\mathfrak{r}\}^c}), 1] = [(\delta_{\mathfrak{p}}^{\{\mathfrak{r}\}^c}), 1]$$

yields that

$$[(b_{\mathfrak{q}}), \beta] \cdot [(\delta_{\mathfrak{q}}^{\{\phi(\mathfrak{r})\}^c}), 1] = [(\delta_{\mathfrak{q}}^{\{\phi(\mathfrak{r})\}^c}), 1]$$

The quotient by $\hat{\mathcal{O}}_R^*(L)$ in the definition of $Y_{L,R}$ means that we can find an $(r_{\mathfrak{q}}) \in \hat{\mathcal{O}}_R^*(L)$ such that $\vartheta_{L^{\mathrm{ab}}(R)}(r_{\mathfrak{q}}) = \beta$ and $(b_{\mathfrak{q}}) \cdot (\delta_{\mathfrak{q}}^{\{\phi(\mathfrak{r})\}^c}) = (r_{\mathfrak{q}}) \cdot (\delta_{\mathfrak{q}}^{\{\phi(\mathfrak{r})\}^c})$. In particular, this means that $b_{\mathfrak{q}} = r_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \phi(\mathfrak{r})$. So $s_L(\phi(\mathfrak{r})) = (r_{\mathfrak{q}}^{-1} b_{\mathfrak{q}})$ is a valid split. □

Proposition 5.2.6. *Take s_L as described in the previous proposition. Then for all $\mathfrak{p} \in \mathcal{D}_S^+(K)$*

$$\psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p}) = \vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(\mathfrak{p}).$$

Proof. We have that $\mathfrak{p}$ is represented by

$$[s_K(\mathfrak{p}), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p})^{-1}] = [s_K(\mathfrak{p}), 1] \cdot [1, \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p})^{-1}].$$

Choose a $\beta \in \mathrm{Gal}(L^{\mathrm{ab}}(R)/L)$ so that $\Phi([1, \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p})^{-1}]) = [1, \beta]$. By the previous proposition we have that $\Phi(\mathfrak{p}) = [s_L \circ \phi(\mathfrak{p}), \beta]$. But we also have that $\phi(\mathfrak{p})$ is represented in $Y_{L,R}$ by $[s_L \circ \phi(\mathfrak{p}), \vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(\mathfrak{p})^{-1}]$. Once again the balancing by $\hat{\mathcal{O}}_R^*$ implies that $\vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(\mathfrak{p})^{-1} = \beta$. The proposition then follows. □

This completes the proof that an algebraic equivalence implies an S-R Reciprocity Isomorphism.

5.2.2 From S-R Reciprocity Isomorphism back to Topological Dynamics

So far we have shown that an algebraic equivalence between $\mathcal{D}_S^+(K) \curvearrowright Y_{K,S}$ and $\mathcal{D}_R^+(R) \curvearrowright Y_{L,R}$ implies an S-R reciprocity isomorphism. We will now recover an algebraic equivalence from the S-R Reciprocity Isomorphism. Once again we will make heavy use of the fact that the monoid of effective divisors is dense in $Y_{K,S}$. Let s_K, s_L, ϕ, and ψ be as described in Definition 5.5 and let

$$\iota : \mathcal{D}_S^+(K) \to Y_{K,S} \text{ by } D \mapsto [s_K(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)^{-1}]$$

be the map embedding the effective S-divisors into $Y_{K,S}$. Define a function

$$\Psi : \iota(\mathcal{D}_S^+(K)) \to Y_{L,R} \text{ by } [s_K(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)^{-1}] \mapsto [s_L \circ \phi(D), \psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)^{-1}].$$

Since ϕ and ψ are both isomorphisms we can see that Ψ is a bijection onto its image $\iota(\mathcal{D}_R^+(L))$.

Proposition 5.2.7. *The extension $\tilde{\Psi} : Y_{K,S} \to Y_{L,R}$ of Ψ given by*

$$\tilde{\Psi}([(a_{\mathfrak{p}}), \alpha]) = \lim_{j \to \infty} \Psi([s_K(D_j), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1}])$$

is well defined, where $D_j \in \mathcal{D}_S^+(K)$ is any sequence such that $[s_K(D_j), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1}]$ converges to $[(a_{\mathfrak{p}}), \alpha]$.

Proof. Since by Lemma 3.1.13 the subspace $\iota(\mathcal{D}_S^+(K))$ is dense in $Y_{K,S}$, so for any $[(a_{\mathfrak{p}}), \alpha] \in Y_{K,S}$ we can choose a sequence $D_j \in \mathcal{D}_S^+(K)$ such that

$$\lim_{j \to \infty} [s_K(D_j), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1}] = [(a_{\mathfrak{p}}), \alpha].$$

As in the proof of Lemma 3.1.13, we can write $(a_{\mathfrak{p}}) = (r_{\mathfrak{p}})\cdot(u_{\mathfrak{p}})$ where $(r_{\mathfrak{p}}) \in \hat{\mathcal{O}}_S^*$ and each $u_{\mathfrak{p}} = p_{\mathfrak{p}}(s_K(\mathfrak{p})^{n_{\mathfrak{p}}})$ for $n_{\mathfrak{p}} = v_{\mathfrak{p}}(a_{\mathfrak{p}})$ and $p_{\mathfrak{p}} : \mathbb{A}_S^* \to \mathcal{O}_{\mathfrak{p}}^*$ the projection to the $\mathfrak{p}$ component. By definition of $Y_{K,S}$ we have that

$$[(a_{\mathfrak{p}}), \alpha] = [(u_{\mathfrak{p}}), \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})\alpha].$$

It is clear that the sequence $s_K(D_j)$ converges to $(u_{\mathfrak{p}})$ in $\hat{\mathcal{O}}_S$, moreover that $s_L \circ \phi(D_j)$ converges to $(u'_{\mathfrak{q}}) \in \hat{\mathcal{O}}_R$ where $u'_{\mathfrak{q}} = s_L(\mathfrak{q})^{n_{\phi^{-1}(\mathfrak{q})}}$ and s_L is the split given by our S-R reciprocity isomorphism. Say that

$$\lim_{j\to\infty} \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1} = \alpha'$$

then we have that

$$[(a_{\mathfrak{p}}), \alpha] = [(u_{\mathfrak{p}}), \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})\alpha] = [(u_{\mathfrak{p}}), \alpha'].$$

The quotient by $\hat{\mathcal{O}}_S^*$ in $Y_{K,S}$ means that there exists an $(s_{\mathfrak{p}}) \in \hat{\mathcal{O}}_S^*$ such that $(u_{\mathfrak{p}} s_{\mathfrak{p}}^{-1}) = (u_{\mathfrak{p}})$ and $\alpha' = \alpha \cdot \vartheta_{K^{\mathrm{ab}}(S)}(s_{\mathfrak{p}}) \cdot \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})$. The condition that $(u_{\mathfrak{p}} s_{\mathfrak{p}}^{-1}) = (u_{\mathfrak{p}})$ means that $s_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in \mathrm{supp}(a_{\mathfrak{p}})$. Note that in $Y_{L,R}$ the sequence $[s_L \circ \phi(D_j), \psi\big(\vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1}\big)]$ converges. By continuity of ψ we have that

$$\lim_{j\to\infty} [s_L \circ \phi(D_j), \psi\big(\vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)^{-1}\big)] = [(u'_{\mathfrak{q}}), \psi(\alpha')].$$

Since an S-R reciprocity isomorphism provides that $\psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ i(\mathcal{O}_{\mathfrak{p}}^*) = \vartheta_{L^{\mathrm{ab}}(R)} \circ i(\mathcal{O}_{\phi(\mathfrak{p})}^*)$ we have that $\psi(\alpha') = \psi(\alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))\vartheta_{L^{\mathrm{ab}}(R)}(s'_{\mathfrak{q}})$ for some

$(s'_{\mathfrak{q}}) \in \mathcal{O}^*_R$ with $s'_{\mathfrak{q}} = 1$ when $\mathfrak{q} \in \phi(\mathrm{supp}(a_{\mathfrak{p}})) = \mathrm{supp}(u'_{\mathfrak{q}})$. Hence we have that

$$(s'_{\mathfrak{q}}) \cdot \big((u'_{\mathfrak{q}}), \psi(\alpha')\big) = ((u'_{\mathfrak{q}}), \psi(\alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})))$$

which finally means that

$$\lim_{j\to\infty}[s_L \circ \phi(D_j), \psi\big(\vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D_j)\big)^{-1}] = [(u'_{\mathfrak{q}}), \psi(\alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))].$$

Since $(u'_{\mathfrak{q}})$ and $(r_{\mathfrak{p}})$ only depend on $(a_{\mathfrak{p}})$ this shows that the map is well-defined. $\square$

Proposition 5.2.8. *The function $\tilde{\Psi} : Y_{K,S} \to Y_{L,R}$ extending Ψ is continuous.*

Proof. Let $[(a^j_{\mathfrak{p}}), \alpha_j] \in Y_{K,S}$ be a sequence converging to $[(a_{\mathfrak{p}}), \alpha]$. As before, define $(r^j_{\mathfrak{p}})$ and $(u^j_{\mathfrak{p}})$ so that $(a^j_{\mathfrak{p}}) = (r^j_{\mathfrak{p}}) \cdot (u^j_{\mathfrak{p}})$ with $(r^j_{\mathfrak{p}}) \in \hat{\mathcal{O}}^*_S$ and $u^j_{\mathfrak{p}} = p_{\mathfrak{p}}(s_K(\mathfrak{p})^{n^j_{\mathfrak{p}}})$ for $n^j_{\mathfrak{p}} = v_{\mathfrak{p}}(a^j_{\mathfrak{p}})$. We also define $(u_{\mathfrak{p}}) \in \hat{\mathcal{O}}_S(K)$ and $(r_{\mathfrak{p}}) \in \hat{\mathcal{O}}^*_S(K)$ in the same manner so that $(a_{\mathfrak{p}}) = (u_{\mathfrak{p}})(r_{\mathfrak{p}})$. By the previous proposition we have that

$$\tilde{\Psi}[(a^j_{\mathfrak{p}}), \alpha_j]) = [(u'^j_{\mathfrak{q}}), \psi(\alpha_j\vartheta_{K^{\mathrm{ab}}(S)}(r^j_{\mathfrak{p}}))] \text{ and } \tilde{\Psi}[(a_{\mathfrak{p}}), \alpha]) = [(u'_{\mathfrak{q}}), \psi(\alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))]$$

where $u'^j_{\mathfrak{q}} = p_q(s_L(\mathfrak{q})^{n^j_{\phi^{-1}(\mathfrak{q})}})$ and $u'_{\mathfrak{q}} = s_L(\mathfrak{q})^{n_{\phi^{-1}(\mathfrak{q})}}$. We want to show that

$$\lim_{j\to\infty}[(u'^j_{\mathfrak{q}}), \psi(\alpha_j\vartheta_{K^{\mathrm{ab}}(S)}(r^j_{\mathfrak{p}}))] = [(u'_{\mathfrak{q}}), \psi(\alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))].$$

Note that $(u^j_{\mathfrak{p}})$ converges to $(u_{\mathfrak{p}})$ and $(u'^j_{\mathfrak{q}})$ to $(u'_{\mathfrak{q}})$. Since $\hat{\mathcal{O}}_S(K)$ and $\hat{\mathcal{O}}^*_S(K)$ are both compact, we may assume that $\lim_{j\to\infty}(a^j_{\mathfrak{p}}) = (\tilde{a}_{\mathfrak{p}})$, and $\lim_{j\to\infty}(r^j_{\mathfrak{p}}) = (\tilde{r}_{\mathfrak{p}})$ by using convergent sub-sequences and re-labelling the sequences. We also have that $\lim_{j\to\infty}\alpha_j = \tilde{\alpha}$. Altogether this means that $[(u_{\mathfrak{p}}), \alpha\vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}})] = [(u_{\mathfrak{p}}), \tilde{\alpha}\vartheta_{K^{\mathrm{ab}}(S)}(\tilde{r}_{\mathfrak{p}})]$. Consequently, we may chose an $(s_{\mathfrak{p}}) \in \hat{\mathcal{O}}^*_S(K)$ with $s_{\mathfrak{p}} = 1$

when $\mathfrak{p} \in \operatorname{supp}(u_{\mathfrak{p}})$ to act on $((u_{\mathfrak{p}}), \alpha \cdot \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))$ and show that $\alpha \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}} s_{\mathfrak{p}}^{-1}) = \tilde{\alpha} \vartheta_{K^{\mathrm{ab}}(S)}(\tilde{r}_{\mathfrak{p}})$. Thus we have

$$\lim_{j \to \infty} \psi(\alpha_j \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}^j)) = \psi(\tilde{\alpha} \vartheta_{K^{\mathrm{ab}}(S)}(\tilde{r}_{\mathfrak{p}})) = \psi(\alpha \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}} s_{\mathfrak{p}}^{-1}))$$

yielding that

$$\lim_{j \to \infty} [(u'^j_{\mathfrak{q}}), \psi(\alpha_j \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}^j))] = [(u'_{\mathfrak{q}}), \vartheta_{L_R}(s'_{\mathfrak{q}}) \psi(\alpha \vartheta_{K^{\mathrm{ab}}(S)}(r_{\mathfrak{p}}))]$$

where $s'_{\mathfrak{q}} = 1$ when $\mathfrak{q} \in \phi(\operatorname{supp}(a_{\mathfrak{p}})) = \operatorname{supp}(u'_{\mathfrak{q}})$, as in the previous proposition. Since $(u'_{\mathfrak{q}} s'_{\mathfrak{q}}) = (u'_{\mathfrak{q}})$, the result follows. $\square$

Note. *It is known in [4], Chapter II, Section 4.2 that there is a unique uniform structure on a compact space X that is compatible with its topology. Moreover, that if $f : A \to Y$ is a function from a dense subspace $A \subset X$ to a complete, Hausdorff uniform space Y then f has a continuous extension to an $\tilde{f} : X \to Y$ if and only if f is uniformly continuous. Since $Y_{K,S}$ is compact we have shown that Ψ is uniformly continuous with respect to the uniform structure resulting from dense subspace, as the reader may have suspected.*

Proposition 5.2.9. *The continuous map $\tilde{\Psi} : Y_{K,S} \to Y_{L,R}$ is an isomorphism of topological monoids.*

Proof. The map $\Psi : \iota(\mathcal{D}_S^+(K)) \to \iota(D_R^+(L))$ is a monoid morphism by construction. The construction of Ψ can be repeated with ϕ^{-1} and ψ^{-1} to yield a new continuous map Ψ^{-1}. It is clear that

$$\begin{aligned}\Psi^{-1} \circ \Psi([s_K(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)]) &= \Psi^{-1}([s_L \circ \phi(D), \psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)^{-1}] \\ &= [s_K(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)]\end{aligned}$$

and

$$\begin{aligned}\Psi \circ \Psi^{-1}[s_L(F), \vartheta_{L^{\mathrm{ab}}(R)} \circ s_L(F)]) &= \Psi([s_K \circ \phi^{-1}(F), \psi^{-1} \circ \vartheta_{L,B} \circ s_L(F)]) \\ &= [s_L(F), \vartheta_{L,B} \circ s_L(F)]\end{aligned}$$

$\square$

Proposition 5.2.10. *We have that $\tilde{\Psi} \circ \iota = \phi$.*

Proof.

$$\begin{aligned}\tilde{\Psi} \circ (D) &= \Psi([s_K(D), \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)]) \\ &= [s_L \circ \phi(D), \psi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(D)] \\ &= [s_L \circ \phi(D), \vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \psi(D)] \\ &= \iota(\phi(D))\end{aligned}$$

$\square$

This concludes the proof of Theorem 5.2.1.

5.3 L-function Isomorphisms

We will prove the following:

Theorem 5.3.1. *The following are equivalent:*

(1) There is an S-R Reciprocity Isomorphism between K and L.

(2) There is a Finite Reciprocity Isomorphism between K and L.

(3) There is an L-function Isomorphism between K and L.

In section 5.3.1 we will show that (1) $\implies$ (2) $\implies$ (3) and in section 5.3.2 we will prove the more difficult case of (3) $\implies$ (1). This section extends the results of [13] to the function field framework described in Chapter 3.

5.3.1 Finite Reciprocity to L-Functions

Lemma 5.3.2. *Assume that we have an S-R reciprocity isomorphism for K and L. Then $N_S(\mathfrak{p}) = N_R(\phi(\mathfrak{p}))$.*

Proof. Recall that $N_S(\mathfrak{p}) = \#(\mathcal{O}_S/\mathfrak{p}) = \#(\mathcal{O}_\mathfrak{p}/(\pi_\mathfrak{p})) = q^f$ for some $f \in \mathbb{N}$. Prop. 5.3 of Chapter 2 of [27] tells us that

$$\mathcal{O}_\mathfrak{p}^* \cong \mu_{q^f-1} \times U^{(1)}.$$

Now our assumption of an S-R reciprocity isomorphism means that $\psi \circ \vartheta_K^{\mathrm{ab}}(S) \circ i(\mathcal{O}_\mathfrak{p}) = \vartheta_{L^{\mathrm{ab}}(R)} \circ i \circ (\mathcal{O}_{\phi(\mathfrak{p})}^*)$. Combined with Prop. 3.1.6 we have that $\mathcal{O}_\mathfrak{p}^* \cong \mathcal{O}_{\phi(\mathfrak{p})}^*$. Altogether this tells us that ϕ preserves the number q^f associated to a prime $\mathfrak{p}$ as $q^f - 1$ is the cardinality of the torsion part of $\mathcal{O}_\mathfrak{p}$. □

Proposition 5.3.3. *If we have an S-R Reciprocity Isomorphism between K and L then we also have a Finite Reciprocity Isomorphism.*

Proof. Let K' be a finite extension of K fixed by a subgroup $N \subset \mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$. By Lemma 2.2.17 we know that a prime $\mathfrak{p}$ of K is unramified in K' if and only if $\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^{*}) \subseteq N$. So the S-R reciprocity isomorphism implies that $\phi(\mathfrak{p})$ is unramified in L' and symmetry give us the required bijection. Additionally,

$$\vartheta_{K^{\mathrm{ab}}(S)}(s_K(\mathfrak{p})) \equiv \mathrm{Frob}_{K'/K}(\mathfrak{p}) \mod N$$

regardless of the choice of split s_K. So $\psi(\mathrm{Frob}_{K'/K}(\mathfrak{p})) = \mathrm{Frob}_{L'/L}(\phi(\mathfrak{p}))$ follows. □

Proposition 5.3.4. *If we have an S-R Reciprocity Isomorphism between K and L then we also have an L-function Isomorphism.*

Proof. This is a straightforward application of the previous two propositions and the definitions. As noted before we can write the L function as

$$L_{K,\chi}(z) = \prod_{\mathfrak{p} \in U(\chi)} \big(1 - \chi\big(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p})\big) N_S(\mathfrak{p})^{-z}\big)^{-1}$$

The previous lemma tells us that ϕ is a bijection between the primes unramified in K_χ and $L_{\hat{\psi}(\chi)}$ id est, $\phi(U(\chi)) = U(\hat{\psi}(\chi))$. Additionally it tells us that

$$\hat{\psi}(\chi)(\mathrm{Frob}_{L_{\hat{\psi}(\chi)}/L}(\phi(\mathfrak{p}))) = \chi(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p}))$$

We have also shown that $N_S(\mathfrak{p}) = N_R(\phi(\mathfrak{p}))$. So it follows that

$$\begin{aligned}
L_{K,\chi}(z) &= \prod_{\mathfrak{p}\in U(\chi)} \left(1 - \chi\big(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p})\big) N_S(\mathfrak{p})^{-z}\right)^{-1} \\
&= \prod_{\phi(\mathfrak{p})\in U(\hat{\psi}(\chi))} \left(1 - \hat{\psi}(\chi)(\mathrm{Frob}_{L_{\hat{\psi}(\chi)}/L}(\phi(\mathfrak{p}))) N_R(\phi(\mathfrak{p}))^{-z}\right)^{-1} \\
&= \prod_{\mathfrak{q}\in U(\hat{\psi}(\chi))} \left(1 - \hat{\psi}(\chi)\big(\mathrm{Frob}_{L_{\hat{\psi}(\chi)}/L}(\mathfrak{q})\big) N_R(\mathfrak{q})^{-z}\right)^{-1} \\
&= L_{L,\hat{\psi}(\chi)}(z)
\end{aligned}$$

□

5.3.2 And back again

Let us assume that we have global function fields K and L with finite non-empty sets of primes S and R, respectively, together with an isomorphism of topological groups $\psi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathrm{Gal}(L^{\mathrm{ab}}(R)/L)$ such that

$$L_{K,\chi}(z) = L_{L,\hat{\psi}(\chi)}(z) \text{ for all } z \in \mathbb{C}$$

for all characters $\chi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$. To prove that this implies an S-R reciprocity isomorphism we first introduce some notation from [13]:

Objects which have subscripts of N, $< N$ or $\geq N$ are restricted to working with primes $\mathfrak{p}$ (or divisors) with $N_S(\mathfrak{p}) = N$, $N_S(\mathfrak{p}) < N$, $N_S(\mathfrak{p}) \geq N$ (or with $N_R(\mathfrak{q}) = N$, et cetera, where appropriate). For instance

$$L_{K,\chi,<N}(s) = \prod_{\mathfrak{p}\in U_{<N}(\chi)} \left(1 - \chi(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p})) N_S(\mathfrak{p})^{-s}\right)^{-1}$$

and

$$U_{<N}(\chi) = \{\mathfrak{p} : \mathcal{S}_K : \chi|_{\vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}^*_{\mathfrak{p}})} = 1 \text{ and } N_S(\mathfrak{p}) < N\}$$

The key to the proof is to construct a bijection of primes with certain properties from which we can derive an S-R reciprocity isomorphism. Namely,

Proposition 5.3.5. *Assume that for all natural numbers N there exits a bijection $\phi_N : (\mathcal{S}_K \backslash S)_N \to (\mathcal{S}_L \backslash R)_N$ such that for all characters $\chi : Gal(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ and all $\mathfrak{p} \in \mathcal{S}_K \setminus S$ we have that $\chi(\mathfrak{p}) = \hat{\psi}(\chi) \circ \phi_N(\mathfrak{p})$. Then we have an S-R reciprocity isomorphism between K and L.*

Proof. Let $\phi : \mathcal{S}_K \setminus S \to \mathcal{S}_L \setminus L$ be the map defined by $\phi(\mathfrak{p}) := \phi_{N_S(\mathfrak{p})}(\mathfrak{p})$. This map is a bijection that satisfies $\chi(\mathfrak{p}) = \hat{\psi}(\chi) \circ \phi_N(\mathfrak{p})$ for $\mathfrak{p} \in \mathcal{S}_K \setminus S$ and any character χ. The map ϕ can be extended to a monoid isomorphism $\mathcal{D}^+_S(K) \cong \mathcal{D}^+_R(L)$. Moreover, if $\mathfrak{p} \in U(\chi)$, then by Lemma 2.2.16 $\mathfrak{p}$ is unramified in K_χ. We then have that $\chi(\mathfrak{p}) = \chi(\mathrm{Frob}_{K_\chi/K}(\mathfrak{p})) \neq 0$ by definition and so $\hat{\psi}(\chi)(\mathrm{Frob}_{L_{\hat{\psi}(\chi)}/L}(\phi(\mathfrak{p})) \neq 0$ as well, id est $\mathfrak{p} \in U(\chi) \iff \phi(\mathfrak{p}) \in U(\hat{\psi}(\chi))$. By Lemma 2.2.17 we have that

$$\begin{aligned}
&\psi \circ \vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}^*_{\mathfrak{p}}) \\
&= \psi\Big(\bigcap_{\chi:\mathfrak{p}\in U(\chi)} \ker\chi\Big) \\
&= \bigcap_{\chi:\mathfrak{p}\in U(\chi)} \psi(\ker\chi) \\
&= \bigcap_{\hat{\psi}(\chi):\phi(\mathfrak{p})\in U(\hat{\psi}(\chi))} \ker\hat{\psi}(\chi) \\
&= \vartheta_{L^{\mathrm{ab}}(R)}(\mathcal{O}^*_{\phi(\mathfrak{p})})
\end{aligned}$$

the last equality comes from the fact that ψ is an isomorphism, which means that $\hat{\psi}$ and $\hat{\psi}^{-1}$ are mutually inverse maps on their respective character groups. Moreover, by assumption we have that $\chi(\mathfrak{p}) = \hat{\psi}(\chi) \circ \phi_N(\mathfrak{p})$ for all characters χ and all $\mathfrak{p} \in \mathcal{S}_K \setminus S$. In particular for $\mathfrak{p} \in U(\chi)$, which means that

$$\chi \circ \vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p}) = \hat{\psi}(\chi)\big(\vartheta_{L_K} \circ s_L \circ \phi(\mathfrak{p})\big).$$

for all splits s_K and s_L. But this means that

$$\vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p}) \equiv \psi^{-1}\big(\vartheta_{L_K} \circ s_L \circ \phi(\mathfrak{p})\big) \mod \ker \chi$$

This is true for all χ, in particular all χ such that $\mathfrak{p} \in U(\chi)$, allowing us to take the quotient of the intersection of all such $\ker \chi$, which by Lemma 2.2.17 means that

$$\vartheta_{K^{\mathrm{ab}}(S)} \circ s_K(\mathfrak{p}) \equiv \psi^{-1}\big(\vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(\mathfrak{p})\big) \mod \vartheta_{K^{\mathrm{ab}}(S)}(\mathcal{O}_{\mathfrak{p}}^*).$$

Id est, for all $\mathfrak{p} \in \mathcal{S}_K \setminus S$ there exists an element, say $r_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^*$ such that

$$\vartheta_{K^{\mathrm{ab}}(S)}(s_K(\mathfrak{p})r_{\mathfrak{p}}) = \phi^{-1}\big(\vartheta_{L^{\mathrm{ab}}(R)} \circ s_L \circ \phi(\mathfrak{p})\big).$$

Therefore we can define a new split $s'_K(\mathfrak{p}) := s_K(\mathfrak{p})r_{\mathfrak{p}}$ to satisfy the conditions of an S-R reciprocity isomorphism. □

To construct such a bijection ϕ_N we will proceed inductively. For $N = 1$ there is nothing to show. For the remainder of this section we will assume that we have such a ϕ_N for all $N \leq M$. For $N > M$ the construction of such a bijection will result from the characters $\chi_{\mathfrak{p}}$ with distinguished prime $\mathfrak{p}'$ from Prop. 3.1.7. We will use this to construct (in Prop. 5.3.9) the desired bijection of primes of S-norm N. To develop this bijection, we will need to show that if $\chi_{\mathfrak{p}}$ is a character that distinguishes $\mathfrak{p}$ then $\hat{\psi}(\chi_{\mathfrak{p}})$ distinguishes $\phi(\mathfrak{p})$. This is done in Lemma 5.3.6.

Fix $k \in \mathbb{N}$ with $k > 2$ and let $\zeta := e^{\frac{2\pi i}{k}}$. We will use the following definitions: $\ker_N \chi$ denotes the set of all primes $\mathfrak{p} \notin S$ with $N_S(\mathfrak{p}) = N$ and $\chi(\mathfrak{p}) = 1$; we do not consider primes with $\chi(\mathfrak{p}) = 0$. Let $u_N(\chi) = \#U_N(\chi)$, $k_N(\chi) = \# \ker_{s_K} \chi$, $c_N = \#(\mathcal{S}_{K,N} \setminus S)$, and finally

$$\Xi_K^1 := \{\chi \in \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{ab}}(S)/K), \mathbb{T}) : u_N(\chi) = k_N(\chi) = c_N\}$$

If $k_N(\chi) = u_N(\chi) - 1$ then there exists a unique prime $\mathfrak{p}_\chi$ of norm N that, when evaluated by χ, has value of neither 0 nor 1. So we also define

$$\Xi_K^2 := \{\chi \in \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{ab}}(S)/K), \mathbb{T}) : u_N(\chi) = c_N,\ k_N(\chi) = c_N - 1,\ \chi(\mathfrak{p}_\chi) = \zeta\}$$

Note that by Prop. 3.1.7 Ξ_K^2 is non-empty. Moreover, if $\chi \in \Xi_K^2$ then $\chi^k \in \Xi_K^1$, so Ξ_K^1 is non-empty as well. Since the characters in Ξ_K^2 carry a distinguished prime, in order to construct our bijection we will need to show:

Proposition 5.3.6. $\hat{\psi}(\Xi_K^2) = \Xi_L^2$

This assertion becomes more clear after some preliminary results. We may assume that we have a

$$\phi_{<N} : \mathcal{S}_{K,<N} \setminus S \to \mathcal{S}_{L,<N} \setminus R$$

obtained from combining each ϕ_M described previously. Note that we have already shown that:

$$N_S(\mathfrak{p}) = N_R(\phi_{<N}(\mathfrak{p})$$

$$U_{<N}(\hat{\psi}(\chi)) = \phi_{<N}\big(U_{<N}(\chi)\big), \text{ and}$$

$$\chi(\mathfrak{p}) = \hat{\psi}(\chi)(\phi_{<N}(\mathfrak{p}))$$

for $\mathfrak{p} \in \mathcal{S}_{K,<N} \setminus S$. From this it follows that $L_{K,\chi,<N}(z) = L_{L,\hat{\psi}(\chi),<N}(z)$. Since by assumption we have that $L_{K,\chi} = L_{L,\hat{\psi}(\chi)}$ we also have that $L_{K,\chi,\geq N}(z) = L_{L,\hat{\psi}(\chi),\geq N}(z)$. We will use this to show that:

Lemma 5.3.7. $\hat{\psi}(\Xi_K^1) = \Xi_L^1$

Proof. For any character $\chi : \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \to \mathbb{T}$ define

$$\mathscr{X}_N(\chi) := \sum_{\mathfrak{p} \in \mathcal{S}_{K,N} \setminus S} \chi(\mathfrak{p}) = \sum_{\mathfrak{p} \in U_N(\chi)} \chi(\mathfrak{p})$$

Note that $\chi \in \Xi_K^1 \iff \mathscr{X}_N(\chi) = c_N$. So it will suffice to show that $\mathscr{X}_N(\chi) = \mathscr{X}_N(\hat{\psi}(\chi))$. Let $\mathcal{D}^+_{S,\geq N}(K)$ be the sub-monoid of $\mathcal{D}^+_S(K)$ generated by $\mathcal{S}_{K,\geq N} \setminus S$. We note that we can write $L_{K,\chi,\geq N}$ as

$$L_{K,\chi,\geq N}(z) = \prod_{\mathfrak{p} \in U(\chi)_{\geq N}} \big(1 - \chi(\mathfrak{p}) N_S(\mathfrak{p})^{-z}\big)^{-1} = \sum_{D \in \mathcal{D}^+_{S,\geq N}(K)} \chi(D) N_S(D)^{-z}$$

By factoring out primes of the same S-norm we may write

$$L_{K,\chi,\geq N}(z) = \sum_{M \geq N} \Big(\sum_{D \in \mathcal{D}^+_{S,\geq N}(K) \cap (\mathcal{S}_{K,M} \setminus S)} \chi(D) \Big) M^{-z}$$

Since $L_{K,\chi,\geq N}(z) = L_{L,\hat{\psi}(\chi),\geq N}(z)$ we must have that the coefficients of M^{-z}

are equal, so we have

$$\sum_{D\in\mathcal{D}^+_{S,\geq N}(K)\cap(\mathcal{S}_{K,M}\setminus S)} \chi\circ s_K(D) = \sum_{F\in\mathcal{D}^+_{R,\geq N}(L)\cap(\mathcal{S}_{L,M}\setminus R)} \hat{\psi}(\chi)\circ s_L(F)$$

But since

$$\mathcal{D}^+_{S,\geq N}(K)\cap(\mathcal{S}_{K,M}\setminus S) = \mathcal{S}_{K,M}\setminus S$$

and

$$\mathcal{D}^+_{R,\geq N}(L)\cap(\mathcal{S}_{L,M}\setminus R) = \mathcal{S}_{L,M}\setminus R$$

we have that $\mathscr{X}_N(\chi) = \mathscr{X}_N(\hat{\psi}(\chi))$. □

Note that the above proof also shows that

$$\#(\mathcal{S}_{K,N}\setminus S) = \#(\mathcal{S}_{L,N}\setminus R)$$

by calculating $\mathscr{X}(\chi)$ for $\chi = 1$. We also require the following technical lemma and roots of unity:

Lemma 5.3.8. *([13] Lemma 6.6) For $k\geq 3$ let μ_k denote the k^{th} roots of unity and let $\zeta = e^{\frac{2\pi i}{k}}$. If $a_i\in\mu_k\cup\{0\}$ for $i = 1,\dots n$ are such that $\sum_i a_i = n-1+\zeta$ then there is a $1\leq j\leq n$ such that $a_j = \zeta$, and $a_i = 1$ for all $i\neq j$.*

Proof. If $k = 3$ then ζ is the only member of $\mu_k\cup\{0\}$ with positive imaginary part, and the result is obvious. For $k > 3$ define $R := \max\operatorname{Re}(\mu_k - \{1\}) = \operatorname{Re}(\zeta)$ and f to be the number of a_i not equal to 1. Since $0\leq R < 1$ we have that $n-1+R = \operatorname{Re}(\sum_i a_i)\leq n-f+fR$ id est that $R-1\leq f(R-1)$, which implies that $f\leq 1$.

Proof. (Of Prop. 5.3.6) For any character $\chi\in\Xi^2_K$ it is clear that $\chi^k\in\Xi^1_K$.

So then $\hat{\psi}(\chi)^k \in \Xi_L^1$ and for any $\mathfrak{q} \in \mathcal{S}_L \setminus R$ we have that $\hat{\psi}(\chi)(\mathfrak{q}) \in \mu_k \cup \{0\}$. We also have that $\mathscr{X}_N(\hat{\psi}(\chi)) = \mathscr{X}_N(\chi) = c_N - 1 + \zeta$. So the previous lemma implies that there exists a unique prime $\mathfrak{q}_{\hat{\psi}(\chi)}$ such that $\hat{\psi}(\chi)(\mathfrak{q}_{\hat{\psi}(\chi)}) = \zeta$ and that $\hat{\psi}(\chi)(\mathfrak{q}) = 1$ for all other $\mathfrak{q} \neq \mathfrak{q}_{\hat{\psi}(\chi)}$ with $N_R(\mathfrak{q}) = N$. Hence $\hat{\psi}(\Xi_K^2) \subset \Xi_L^2$. Since ψ is an isomorphism we have equality. $\square$

By Prop. 3.1.7 we have that for every prime $\mathfrak{p} \in \mathcal{S}_{K,N} \setminus S$ there exists a character $\chi \in \Xi_K^2$ whose distinguished prime $\mathfrak{p}_\chi$ is equal to $\mathfrak{p}$. The previous lemma provides a mapping between characters with distinguished primes, so to every prime $\mathfrak{p}_\chi \in \mathcal{S}_{K,N} \setminus S$ we may associated a prime $\mathfrak{q}_{\hat{\psi}(\chi)} \in \mathcal{S}_{L,N} \setminus R$. We have not yet shown that this association is a well-defined function; if two characters χ and χ' of $\mathrm{Gal}(K^{\mathrm{ab}}(S)/K)$ both have the same distinguished prime $\mathfrak{p}$, we have not yet shown that $\hat{\psi}(\chi)$ and $\hat{\psi}(\chi')$ necessarily share the same distinguished prime in $\mathcal{S}_{L,N} \setminus R$. Hence the next proposition:

Proposition 5.3.9. *The map $\phi_N : \mathcal{S}_{K,N} \setminus S \to \mathcal{S}_{L,N} \setminus R$ by $\mathfrak{p}_\chi \to \mathfrak{q}_{\hat{\psi}(\chi)}$ is a well-defined bijection between such that for every $\chi \in \mathrm{Hom}(Gal(K^{\mathrm{ab}}(S)/K), \mathbb{T})$ and $\mathfrak{p} \in \mathcal{S}_{K,N} \setminus S$ we have that $\chi(\mathfrak{p}) = \hat{\psi}(\chi)(\phi_N(\mathfrak{p}))$.*

Proof. Let $\chi, \chi' \in \Xi_K^2$ be such that $\mathfrak{p}_\chi = \mathfrak{p}_{\chi'}$ and $\mathfrak{q}_{\hat{\psi}(\chi)} \neq \mathfrak{q}_{\hat{\psi}(\chi')}$. We have that

$$\mathscr{X}_N(\chi \cdot \chi') = \sum_{\mathfrak{p} \in \mathcal{S}_{K,N} \setminus S} \chi(\mathfrak{p})\chi'(\mathfrak{p}) = c_N - 1 + \zeta^2$$

since both χ and χ' take 1 or 0 on all but the same single prime in $\mathcal{S}_{K,N}$ by definition, and $\chi(\mathfrak{p}_\chi) = \chi'(\mathfrak{p}_\chi) = \zeta$. On the other hand,

$$\mathscr{X}_N(\hat{\psi}(\chi \cdot \chi')) = \sum_{\mathfrak{q} \in \mathcal{S}_{L,N}} \hat{\psi}(\chi)(\mathfrak{q}) \cdot \hat{\psi}(\chi')(\mathfrak{q}) = c_N - 2 + 2\zeta$$

This is a contradiction of $\mathscr{X}(\chi) = \mathscr{X}(\hat{\psi}(\chi))$, which was proved in the proof of Lemma 5.3.7, so we conclude that $q_{\hat{\psi}(\chi)} = \mathfrak{q}_{\hat{\psi}(\chi')}$ and ϕ_N is a well-defined map. By considering $\hat{\psi}^{-1}$ we have a well-defined inverse.

For the second part, let χ be an arbitrary character and let $\chi' \in \Xi_K^2$. Let $\mathfrak{p} := \mathfrak{p}_{\chi'}$. Then

$$\mathscr{X}_N(\chi \cdot \chi') = \mathscr{X}_N(\chi) + \chi(\mathfrak{p})(\zeta - 1).$$

Similarly we have that

$$\mathscr{X}_N(\hat{\psi}(\chi \cdot \chi')) = \mathscr{X}_N(\hat{\psi}(\chi)) + \hat{\psi}(\chi)(\phi_N(\mathfrak{p}))(\zeta - 1).$$

Here we are using that $\zeta = \chi'(\mathfrak{p}) = \hat{\psi}(\chi')(\phi_N(\mathfrak{p}))$, which comes from our construction of characters. The proof of Lemma 5.3.7 tells us that $\mathscr{X}_N(\chi \cdot \chi') = \mathscr{X}_N(\hat{\psi}(\chi \cdot \chi'))$ which implies that

$$\chi(\mathfrak{p}) = \hat{\psi}(\chi)(\phi_N(\mathfrak{p}))$$

as desired. $\square$

When combined with the work in Section 5.3.1, propositions 5.3.3, 5.3.4, 5.3.5, and 5.3.9 together complete the proof of Theorem 5.3.1.

Chapter 6

Outlook

Isomorphism of global function fields

This thesis was able to extend many of the results of [13] and [11] to the function field construction of Bost-Connes system described in [30], with one obvious shortcoming: the corresponding statement of Theorem 5.1 for K and L being either both number fields or both function fields for $S = \emptyset$ found in [13] states that we have an isomorphism of fields $K \cong L$ as well. We were unable to prove the same result here.

In the same paper, the authors discuss a certain "incompatibility" of proof techniques for the number field and function field case. When both K and L are number fields, the paper proceeds from and L-function isomorphism to studying the induction of representations of the maximal abelian Galois group of K and L, respectively, to representations of the maximal abelian Galois group of $\mathbb{Q}$. This method of proof fails for global function fields because there no clear replacement for $\mathbb{Q}$; one would need to make an arbitrary choice of a rational function field $\mathbb{F}_q(t)$ that sits inside both K and L. It is clear that our construction suffers from the same problem. When both K and L are function fields, [13] proves that an L-function Isomorphism for $S = \emptyset$ results in an isomorphism $K \to L$ between the function fields. The proof in this case begins with a result from [34]:

Theorem. *(Uchida/Hoshi, Lemmas 8-122 in [34]) Let $\pi_{\mathfrak{p}} : \mathbb{A}_K^* \to K_{\mathfrak{p}}^*$ be the projection from the idele group to the local field. An isomorphism $\Phi : K^* \to L^*$ of the multiplicative groups of the two global fields is the restriction of an isomorphism of fields if and only if there exists a bijection $\phi : \mathcal{S}_K \to \mathcal{S}_L$ such that for all $\mathfrak{p} \in \mathcal{S}_K$ we have that*

(i.) $\Phi(1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}}) = 1 + \phi(\mathfrak{p})\mathcal{O}_{\mathfrak{p}}$ *as sets.*

(ii.) $v_{\phi(\mathfrak{p})} \circ \pi_{\phi(p)} \circ \Phi = v_{\mathfrak{p}} \circ \pi_{\mathfrak{p}}$.

The proof is then to simply show that these conditions are met. In particular, in [13] it is shown that if one has a L-function isomorphism for two global fields K and L then one has that $\ker \vartheta_K \cong \ker \vartheta_L$ as groups. When K and L are function fields and $S = R = \emptyset$, then $\ker \vartheta_K = K^*$ and the result follows.

This technique fails, however, for both number fields and for our construction with non-empty S for the same reason: the kernel of the Artin reciprocity map is too big. For number fields we have that $\ker \vartheta_K = K^* \cdot \overline{\mathcal{O}_+^*(K)}$, where $\mathcal{O}_+^*(K)$ is the group of totally positive units of K embedded in the ideles. In our case we have that $\ker \vartheta_{K_B} = \overline{K^* \cdot K_S^*}$. We may take some solace in the fact that this form of "failure" provides an additional demonstration that our construction in Chapter 3 brings Bost-Connes systems for function fields in a closer analogue with the number field case. Further, it is shown in [31], Theorem 5, that if global function fields K and L are finite geometric extensions of a rational function field $\mathbb{F}_q(t)$ for which there is an L-function isomorphism, then we may proceed to study the induced representations to find a result similar to that in [13]. It is not clear if the our Theorem 5.1 allows us to show a field isomorphism $K \cong L$ without a fixed rational sub-field.

The Deligne-Ribet Monoid

We have referred to the topological monoid $Y_{K,S}$ defined in Chapter 3 without any explanation or justification. We now rectify that situation by giving a rough sketch of a proof that $Y_{K,S}$ is isomorphic to an function field version of the Deligne-Ribet monoid. The Deligne-Ribet monoid $\mathrm{DR}_{\mathbb{K}}$ for a number field $\mathbb{K}$ was first introduced in [16]. It was used in [38] to show that there exists an **arithmetic subalgebra** of the Bost-Connes system of $\mathbb{K}$. In doing so, the author proved that there an isomorphism of topological monoids

$$Y_{\mathbb{K}} := \hat{\mathcal{O}}_K \times \mathrm{Gal}(\mathbb{K}^{\mathrm{ab}}/\mathbb{K}) \Big/ \hat{\mathcal{O}}_{\mathbb{K}}^* \cong \mathrm{DR}_{\mathbb{K}}.$$

Following both [38] and [16] we can define the Deligne-Ribet monoid of a global function field K excluding a finite set of primes S. Let $\mathfrak{f} \in \mathcal{D}_S^+(K)$ and define an equivalence relation $\sim_{\mathfrak{f}}$ on $\mathcal{D}_S^+(K)$ by

$$D \sim_{\mathfrak{f}} F \text{ if there exists an } f \in K^* \cap (1 + \mathcal{F}) \text{ such that } \mathrm{div}(f) = D - F$$

where by $\mathcal{F}$ we mean the ideal in $\mathcal{O}_S(K)$ associated to the divisor $(\mathfrak{f} - F)$ by Theorem 2.1.4. This definition follows the number field definition except in that we use the language of divisors. We then define the quotient $\mathrm{DR}_{\mathfrak{f}} :=$

$\mathcal{D}_S^+(K)/\sim_{\mathfrak{f}}$. One can show that we have a projective system and then define

$$\mathrm{DR}_{K,S} := \varprojlim_{\mathfrak{f}} \mathrm{DR}_f.$$

If $U_{\mathfrak{p}}^n = 1 + \mathfrak{p}^n$ denotes the higher unit group in $\mathcal{O}_{\mathfrak{p}}$ then, following [2], we may define the S-congruence subgroup mod $\mathfrak{f}$:

$$\mathbb{A}_S^{\mathfrak{f}} := \big(\prod_{\mathfrak{p}\in S} K_{\mathfrak{p}}^* \times \prod_{\mathfrak{p}\notin S} U_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{f})}\big) \cap \mathbb{A}^*(K),$$

and the S-ray class group mod $\mathfrak{f}$ as $\mathbb{A}^*(K)/(K^* \cdot \mathbb{A}_S^{\mathfrak{f}})$. By Theorem 2.2.5 (D) there is a unique abelian field extension $K_{\mathfrak{f}}/K$ associated to the S-ray class group mod $\mathfrak{f}$. It is possible to prove that

$$\mathrm{DR}_{\mathfrak{f}} \cong \mathrm{Gal}(K_{\mathfrak{f}}/K)$$

and further that

$$\mathrm{DR}_K^* \cong \mathrm{Gal}(K_B/K).$$

The latter isomorphism follows from recognizing that $K_S^* \cdot K^* \subset (K^* \cdot \mathbb{A}_S^{\mathfrak{f}})$ and $\cap_f K^* \cdot \mathbb{A}_S^f = \overline{K_S^* \cdot^*}$. With this group isomorphism in hand, one can follow [38] to prove that

$$Y_{K,S} \cong \mathrm{DR}_{K,S}.$$

Index of Notation

$(\delta_{\mathfrak{p}}^{P})$ $(\delta_{\mathfrak{p}}^{P}) \in \hat{\mathcal{O}}_S(K)$ such that $\delta_{\mathfrak{p}}^{P} = 0$ when $\mathfrak{p} \notin P$ and $\delta_{\mathfrak{p}}^{P} = 1$ when $\mathfrak{p} \in P$.

K or L Global function field, a finite extension of $\mathbb{F}_q(t)$

K_S^* Diagonal embedding of K^* into $\mathbb{A}_S^*(K)$

N_S S-norm of S-divisors

$N_{\vartheta_K^{\mathrm{ab}}(S)}(P)$ $N_{\vartheta_K}(P) := \vartheta_{K^{\mathrm{ab}}(S)} \circ i(\prod_{\mathfrak{p}\in P} \mathcal{O}_{\mathfrak{p}}^*)$

P^c for a subset $P \subset \mathcal{S}_K \setminus S$, denotes the set $\mathcal{S}_K \setminus (S \cup P)$.

R_x For $x = [(a_{\mathfrak{p}}), \gamma] \in X_{K,S}$, $R_x = \{\mathfrak{p} \in \mathcal{S}_K : a_{\mathfrak{p}} = 0\}$

S Non-empty finite subset of primes of $\mathcal{S}_K$

$X_{K,S}$ The topological monoid $\mathbb{A}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \hat{\mathcal{O}}_S^*$

$Y_{K,S}$ The Deligne-Ribet topological monoid for a function field K excluding primes S; $\hat{\mathcal{O}}_S(K) \times \mathrm{Gal}(K^{\mathrm{ab}}(S)/K) \Big/ \hat{\mathcal{O}}_S^*(K)$.

$\mathbb{A}(K)$ Topological ring of adeles of K

$\mathbb{A}_S(K)$ Topological ring of S-adeles of K; excluding places at S

$\mathcal{D}(K)$ Divisor group of a global function field K

$\mathcal{D}_S(K)$ S-divisors of K; free abelian group generated by $\mathcal{S}_K \setminus S$

$\mathbb{F}_q$ The finite field of $q = p^n$ elements.

Γ_R For a subset $R \subset \mathcal{S}_K$, $\Gamma_R = \{ad(a_\mathfrak{p}) : a \in \overline{K_S^*} \subset \mathbb{A}_S^*,\ a_\mathfrak{p} = 1 \text{ for } \mathfrak{p} \notin R\} \subset \mathcal{D}_S(K)$

$K^{\mathrm{ab}}(S)$ The extension of the global function field K fixed by $\vartheta_K(\overline{K_S^* \cdot K^*})$.

$\mathcal{O}_S(K)$ Ring of S-integers of a global function field.

$\mathcal{O}_\mathfrak{p}$ The valuation ring of $K_\mathfrak{p}$

$\mathcal{O}_\mathfrak{p}(K)$ the ring $\{f \in K : v_\mathfrak{p}(f) \geq 0\}$

ϑ_K The global Artin map associated to K; $\vartheta_K : \mathbb{A}^* \to \mathrm{Gal}(K^{ab}/K)$

$\hat{G}$ Pontryagin dual of a group G; space of characters $\chi : G \to \mathbb{T}$

$\hat{\mathcal{O}}(K)$ $\prod_{\mathfrak{p} \in \mathcal{S}_K} \mathcal{O}_\mathfrak{p}$

$\hat{\mathcal{O}}_S$ $\prod_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}$

$\mathcal{A}_{K,S}$ C^* algebra of the Bost-Connes system for a global function field K excluding primes S

$\mathcal{C}l(K)$ Divisor class group of K

$\mathcal{P}(K)$ Principal divisors of a global function field K

$\mathcal{S}_K$ set of all primes of K

$\mathfrak{p}$ a prime; equivalence class of valuations on a global function field

$\pi_\mathfrak{p}$ A uniformizer for $\mathcal{O}_\mathfrak{p}(K)$, id est, $\pi_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}(K)$

$\deg(\mathfrak{p})$ Degree of the extension of the residue class field of the valuation ring $\mathcal{O}_{\mathfrak{p}}$ of $\mathfrak{p}$ over $\mathbb{F}_q$

$\mathrm{div}(f)$ Principal divisor of $f \in K$

ad The map $ad : \mathbb{A}_S^*(K) \to \mathcal{D}_S(K)$.

$b^{\mathbb{Z}}$ the free abelian group with a single generator, namely, b

$v_{\mathfrak{p}}$ normalized exponential valuation of K associated to the prime $\mathfrak{p}$

support of $(a_{\mathfrak{p}})$ $\{\mathfrak{p} : a_{\mathfrak{p}} \neq 0\}$

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